数理解析研究所講究録
952巻 1996年 161-169



# ON THE NUMBER OF COMPLEX POINTS OF A SURFACE IN AN ALMOST COMPLEX 4-MANIFOLD

東京大 数理科学　　山田　裕一 ( YUICHI YAMADA)

要旨　4次元概複素多様体内にはめ込まれた2次元閉曲面の自己交差数、複素点の個数に関して成り立つ2つの公式を考える。特に、これらが向き付け不能曲面に対してもmod 2 をとることなく成り立つことに興味をもつ。2つの公式とは [Y1'95] で筆者が示した公式（1）と、[W1,2'84] で示された公式（2）である（本文参照)。（2）は [BF'93] により、$\mathbf{C}^2$ 内の有向曲面の場合については、交差理論による別証が与えられた。[Y2] で筆者はそれを一般の場合へ補った（4、5)。そこでは四元数体 $\mathbf{H}$ の体の構造を効率良く利用する。また、2公式の併用で、[Fo'92] のある問題に答えることができる（3)。

Throughout this paper, we will work in the $C^\infty$ category. Let $M$ be a connected oriented 4-manifold, $F$ a closed and connected surface of Euler characteristic $\chi(F)$. We allow that $F$ is non-orientable. For a given immersion $f$ of $F$ into $M$ with only normal crossings, let $e(f)$ be the normal Euler number of it, and let $f_*[F]$ be the element in $H_2(M;\mathbf{Z}_2)$.

## (1) An Extension of Whitney's Congruence.

We are interested in the relation between $e(f)$ and $f_*[F]$.

**Definition 1.** A map $q$ from $H_2(M,\mathbf{Z}_2)$ to $\mathbf{Z}_4$ is $\mathbf{Z}_4$ quadratic iff $q$ satisfies

$$q(\alpha+\beta)\equiv q(\alpha)+q(\beta)+2(\alpha\bullet\beta) \mod 4,$$

where $\bullet$ is ( $Z_2$-valued ) intersection form on $H_2(M;\mathbf{Z}_2)$, and $2\colon \mathbf{Z}_2\to\mathbf{Z}_4$ is the natural embedding.

In [Y1], we extended Whitney's congruence as follows.

For some time, we assume that $M$ is closed and $H_1(M;\mathbf{Z}) = \{0\}$. We will define a $\mathbf{Z}_4$-quadratic map $q$ from $H_2(M;\mathbf{Z}_2)$ to $\mathbf{Z}_4$ as follows. By the assumption $H_1(M;Z)=\{0\}$, the mod 2-reduction map $p_2$ from $H_2(M;\mathbf{Z})$ to $H_2(M;\mathbf{Z}_2)$ is surjective. For a given element $\alpha$ in $H_2(M;\mathbf{Z}_2)$ , we define $q(\alpha)$ by

$$q(\alpha) \equiv \tilde{\alpha} \circ \tilde{\alpha} \mod 4,$$

where $\tilde{\alpha}$ is an element of ${p_2}^{-1}(\alpha)$ and $\circ$ is the intersection form on $H_2(M;\mathbf{Z})$. The well-definedness of $q$ is easy to see, and $q$ is $\mathbf{Z}_4$-quadratic.

**Example 1.** When $M$ is $\mathbf{C}P^2$, $H_2(\mathbf{C}P^2;\mathbf{Z}_2) \cong \mathbf{Z}_2 a$ and $q(0)=0, q(a)=1$.

Our theorem is,

**Theorem 1. [An Extension of Whitney's Congruence]**
*Under the assumption on $M$ above,*

$$e(f) + 2\chi(F) + 2\sharp self(f) \equiv q(f_*[F]) \quad mod\ 4,$$

*where $\sharp self(f)$ is the number of self-intersection points of $f(F)$.*

In general case in which the only assumption on $M$ is its orientability (We assume that $M$ is neither closed nor compact), we have

**Theorem 1'.** *A map which assigns $e(f)+2\chi(F)$ mod 4 to an embedding $F \subset M$ induces a $\mathbf{Z}_4$-quadratic map from $H_2(M;\mathbf{Z}_2)$ to $\mathbf{Z}_4$. We will also call it $q$.*

**Remark 1.** Many researchers study on Whitney's congruence and its extension. (see [A],[L] and [SS])

**(2) Webster's Formula.**

Let $(M,J)$ be an almost complex manifold of real dimension 4. Let $f\colon F \to (M,J)$ be a "generic" immersion, whose definition can be find in ([W2]or[BF]).

A point $x \in F$ is called a complex point of $f$ iff $f_*T_xF = J(f_*T_xF)$ ([Bi]). By the assumption that $f$ is generic, every complex point is isolated. We let $C(f)$ denote the set of all complex points of $f$. We are concerned with the number of complex points of $f$.

In [W1,2], S.M.Webster has shown the next formula by comparing the index sum of zeros of a section $v$ on $TF$ with those of $\pi J f_* v$ on $NF$, where $TF$(and $NF$,respectively) is the tangent (normal) bundle over $F$ and $\pi$ is the projection onto the second factor of $f^*TM = TF \oplus NF$.

**Theorem 2. [Webster's Formula]** ([W1,2])

*Let $(M, J)$ be a complex 2-manifold and $F$ be a closed Surface. We allow that $F$ is non-orientable. For a "generic" immersion $f\colon F \to (M, J)$, we have*

$$e(f) + \chi(F) = \sum_{x \in C(f)} \epsilon(x) \quad \textit{in } \mathbf{Z}.$$

*where $\epsilon(x)$ is a certain index ($\pm 1$) of a point of $C(f)$.*

This formula was studied by many authors from various aspects (see [IO] and its rich references). In [BF], which is the main reference of [Y2], T.Banchoff and F.Farris reproved the formula explicitly in the case in which $F$ is oriented and $M = \mathbf{C}^2$ by applying an elementary intersection theory of a surface and a 2-complex in the Grassmannian $G(2,4)$. In [Y2], we supplemented their method into the general case. In fact, we study the transformation of the G(2,4) ($\cong S^2 \times S^2$) bundle over $M$ explicitly using $\mathbf{H}$, and we develop an intersection theory for non-orientable surfaces without taking modulo 2. In this article, we introduce the former in section (4) and the latter in section (5).

## (3) Totally real non-orientable surface in $\mathbf{C}P^2$.

**Definition 2.** A immersion $f\colon F \to (M, J)$ is called totally real iff $C(f) = \phi$.

Comparing two formulae

(1) $$e(f) + 2\chi(F) + 2\sharp self(f) \equiv q(f_*[F]) \mod 4, \text{ and}$$

(2) $$e(f) + \chi(F) = \sum_{x \in C(f)} \epsilon(x),$$

we have the following.

**Theorem 3. [A Formula for totally real immersion]**

*Let $(M, J)$ be an almost complex 2-manifold and $f\colon F \to (M, J)$ a totally real immersion of a closed Surface. We allow that $F$ is non-orientable. Then*

$$\chi(F) + 2\sharp self(f) \equiv q(f_*[F]) - \sum_{x \in C(f)} \epsilon(x) \qquad \bmod 4.$$

*In particular, if $f\colon F \to (M, J)$ is a totally real embedding, $\chi(F) \equiv q(f_*[F])$* mod 4.

**Example 2.** Totally real embedded (non-orientable) surface $F$ in $\mathbf{C}P^2$ satisfies $\chi(F) \equiv 0 or 1 \mod 4$. This is the answer for the last sentence of [Fo].

**(4) The Transformation of $G(2, 4)$ bundle over $M$.**

This section is a part of [Y2], which is a step to supplement [BF]'s alternative proof to the case in which $M$ is in general.

Let $V$ be an oriented real 4-dimensional vector space with a metric, i.e., a positive definite inner product, and $G(2, V)$ its Grassmannian manifold :

$$G(2, V) = \{H : \text{2-dimensional oriented subspace of } V\}.$$

It is known that $G(2, V)$ is homeomorphic to $S^2 \times S^2$ (see[CS]). We review it. When we take an orthonormal oriented basis $e = \{e_1, e_2, e_3, e_4\}$, an element $H \in G(2, V)$ can be represented by ordered two vectors $a = \sum_{i=1}^{4} a_i e_i$ and $b = \sum_{i=1}^{4} b_i e_i (a_i, b_i \in \mathbf{R})$ which span $H$. We set

$$\begin{aligned} x_1 &= p_{12} + p_{34}, & y_1 &= p_{12} - p_{34}, \\ x_2 &= p_{13} + p_{42}, & y_2 &= p_{13} - p_{42}, \\ x_3 &= p_{14} + p_{23}, & y_3 &= p_{14} - p_{23}, \end{aligned} \quad \text{where} \quad p_{ij} = det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}.$$

Here we note that $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$.

Let $\phi_e(H) = ([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]) \in (\mathbf{R}^3\backslash\{0\})/\mathbf{R}_{>0} \times (\mathbf{R}^3\backslash\{0\})/\mathbf{R}_{>0} \cong S^2 \times S^2$, where $[\ :\ :\ ]$ is the homogeneous coordinate. We note that $\phi_e(H)$ is well-defined, i.e., it does not depend on the choice of $a$ and $b$.

**Remark 2.** Our identification between $G(2,4)$ and $S^2 \times S^2$ is a little different from the historic one ([CS]-[BF]).

From now on, we use the quaternion field $\mathbf{H}$ :

$$\mathbf{H} = \{\, \alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \,|\, \alpha_i \in \mathbf{R}\,(i = 0,1,2,3)\,\}$$

and some standard identification as follows.

$$\begin{aligned}
\mathbf{R}^4 &= \mathbf{H} \quad \text{(naturally)},\\
\mathbf{R}^3 &= \operatorname{Im}\mathbf{H} \;=\; \{\alpha = \alpha_1 i + \alpha_2 j + \alpha_3 k \;|\alpha_i \in \mathbf{R}\;\},\\
\mathbf{C}^2 &= \mathbf{H} \quad \text{by } (z_0, z_1) \leftrightarrow z_0 + z_1 j,\\
S^3 &= \text{the unit sphere of } \mathbf{H},\\
&\qquad \text{which is a Lie group under the quaternionic multiple},\\
S^1 &= \text{the unit circle of } \mathbf{C} \subset \mathbf{H}, \text{ which is an abelian closed subgroup of } S^3,\\
S^2 &= S^3 \cap \operatorname{Im}\mathbf{H} \quad (S^1 \not\subset S^2).
\end{aligned}$$

We also identify $(\mathbf{R}^3\backslash\{0\})/\mathbf{R}_{>0}$ and $S^2$ canonically.

The following proposition is well known.

**Proposition 1.** *We have the following isomorphisms.*

$$\begin{aligned}
\rho &: \frac{S^3 \times S^3}{\pm(1,1)} \longrightarrow SO(4),\\
\rho' &: S^3/\pm 1 \left(\cong \frac{\{(\alpha,\alpha)\,|\alpha \in S^3\}}{\pm(1,1)}\right) \longrightarrow SO(3),\\
\rho'' &: \frac{S^1 \times S^3}{\pm(1,1)} \longrightarrow U(2),
\end{aligned}$$

*where $\rho(\alpha,\beta)(v) = \alpha v \beta^{-1}$ for $v \in \mathbf{H}$. $\rho'(\alpha) = \rho(\alpha,\alpha)$ and $\rho''$ is the restriction of $\rho$.*

When $V$ is equipped with a complex structure $J$, a self linear map which satisfys $J^2 = -id|_V$ and is compatible with the metric of $V$, we take the basis $e = \{e_1, e_2, e_3, e_4\}$ such that $e_2 = Je_1, e_4 = Je_3$.

Under the notation and identification above, We have the follwing lemma.

**Theorem 4. [Explicit Transformation]**
*When $e = \{e_1, e_2, e_3, e_4\}$ and $e' = \{e'_1, e'_2, e'_3, e'_4\}$ are in the following relation,*

$$e'_j = \sum a_{ij} e_i, \qquad A = (a_{ij}) = \rho(\alpha, \beta) \in SO(4),$$

*$\phi_e$ and $\phi_{e'}$ satisfies the commutative diagram bellow.*

$$\begin{array}{ccc} G(2,V) & \xrightarrow{\phi_{e'}} & S^2 \times S^2 \\ \| & & \downarrow \rho'(\alpha)\times\rho'(\beta) \\ G(2,V) & \xrightarrow{\phi_e} & S^2 \times S^2 \end{array}$$

*where $\rho'(\alpha) \times \rho'(\beta) \in SO(3) \times SO(3)$ act on $S^2 \times S^2$ factorwise.*

Proof. This lemma can be proved only by some troublesome calculus. But here we prove it by using quatenionic multiplication.

For an element $H \in G(2,V)$, when we take orthonormal two vectors $a = \sum_{i=1}^{4} a_i e_i$ and $b = \sum_{i=1}^{4} b_i e_i (a_i, b_i \in \mathbf{R})$ which span $H$, and regard them as elements in $\mathbf{H}$:

$$a = a_0 + a_1 i + a_2 j + a_3 k \quad \text{and} \quad b = b_0 + b_1 i + b_2 j + b_3 k,$$

we have $\phi_e(H) = (-a\bar{b}, -\bar{b}a)$ by definition. Here the right-hand side is an element in $S^2 \times S^2$ because $a$ and $b$ are orthonormal, i.e., $|a| = |b| = 1$ and $\mathrm{Re}(a\bar{b}) = \mathrm{Re}(\bar{b}a) = 0$.

Under the other basis system $e'$, $a'$ and $b'$ corresponding to the above $a$ and $b$ satisfy $a' = \alpha^{-1} a \beta$ and $b' = \alpha^{-1} b \beta$.

$$\begin{aligned} \text{Thus } \phi_{e'}(H) &= (-a'\bar{b'}, -\bar{b'}a') \\ &= (-\alpha^{-1} a \beta \overline{\alpha^{-1} b \beta}, -\overline{\alpha^{-1} b \beta} \alpha^{-1} a \beta) \\ &= (-\alpha^{-1} a \bar{b} \alpha, -\beta^{-1} \bar{b} a \beta) \end{aligned}$$

We have the lemma. $\square$

**Remark 3.** When $V$ has a complex structure, we have $A \in U(2)$ (i.e., $\alpha \in S^1$), thus each of $\{i\} \times S^2$ and $\{-i\} \times S^2$ is kept invariant by the transformation. On the orher hand, when $e_1' = e_1$, we have $A \in SO(3)$ (i.e., $\alpha = \beta$), thus each of $\Delta$ and $\bar{\Delta}$ is kept invariant by the transformation, where $\Delta = \{ (X, X) \mid X \in S^2 \}$ and $\bar{\Delta} = \{ (X, -X) \mid X \in S^2 \} \subset S^2 \times S^2$.

In [BF], they has shown the correspondence bellow,

| In $G(2,V)$ | | In $S^2 \times S^2$ |
|---|---|---|
| $G_{e_1} = \{ H \in G(2,V) \mid e_1 \in H \}$ | $\leftrightarrow$ | $\Delta$ |
| $G_{e_1}^{\perp} = \{ H \in G(2,V) \mid e_1 \perp H \}$ | $\leftrightarrow$ | $\bar{\Delta}$ |
| $C = \{ H \in G(2,V) \mid H$ is spun by $a$ and $Ja$ $\}$ | $\leftrightarrow$ | $\{i\} \times S^2$ |
| $\bar{C} = \{ H \in G(2,V) \mid -H$ is spun by $a$ and $Ja$ $\}$ | $\leftrightarrow$ | $\{-i\} \times S^2$ |

Each of $\Delta, \bar{\Delta}, \{i\} \times S^2$ and $\{-i\} \times S^2$ is homeomorphic to $S^2$. We call the union of them $4S^2$. Here we note that $4S^2$ is a 2-boundary as a 2-chain complex.

The conclusion of this section is summerized as follows.

**Lemma.** *When we are given an explicit transformation of $TM$, we can get that of $S^2 \times S^2$ budle over $M$ which is equivarent to the Grassmannian bundle $G(2,TM)$ over $M$ by the explicit transformation lemma.*

*When an almost complex manifold $(M, J)$ has a unit tangent vector field $e_1$, we have the same correspondence as above under $\phi_e$'s. (For example, $G_{e_1(p)} \subset G(2, T_pM)$ is corresponding to $\Delta \in S^2 \times S^2$ under $\phi_{e(p)}$.)*

For an immersed surface $f(F)$, we take a unit vector field $e_1$ around $f(F)$. For example, $e_1 = \frac{\mathrm{grad}\, g}{\|\mathrm{grad}\, g\|}$, where $g$ is a Morse function on $M$ which has no critical point on $f(F)$ and $g \circ f$ is also a Morse function on $F$. Next, we take the generalized Gaussian map $Gf\colon F \to G(2,TM)$ $(x \mapsto f_* T_x F)$. Then $\mathrm{Int}(Gf, 4S^2\text{-bundle}) = 0$ holds, because $4S^2$ is a boundary. Webster's formula follows the equation.

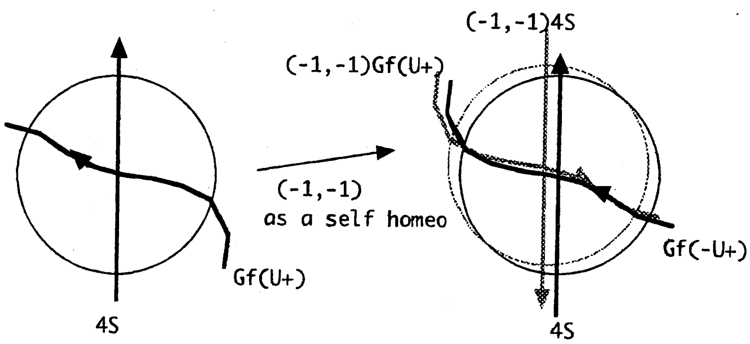
**(5) A non-orientable Surface.**

When $F$ is non-orientable, we can not use $G(2,TM)$ as the image of the general Gaussian map. It may be easy to use the unoriented Grassmannian $G_{\pm}(2,TM) = G(2,TM)/\pm$ and treat the indexes by modulo 2. In fact, each $G_{e_1}$ and $G_{e_1}^{\perp} \subset G_{\pm}(2,TM)$ is homeomorphic to $\mathbf{R}P^2$, and $C \cup \bar{C}$ to a $S^2$. But, here we develop an intersection theory of non-orientable surfaces without taking modulo 2.

**GEOMETRIC PROOF** of Webster formula ($F$ is non-orientable)

We use the following formula : If $\phi_e(H) = (X,Y)$ , then $\phi_e(-H) = (-X,-Y)$. We write this formula as $\phi_e \circ (-1) = (-1,-1) \circ \phi_e$. Here we note that the involution map $(-1,-1)$ of $S^2 \times S^2$ is orientation preserving itself and carry our $4S^2$ to themselves with orientation reversing.

Let $p\colon \hat{F} \to F$ be an orientable double covering of $F$ and $-\colon \hat{F} \to \hat{F}$ the involution associate to the covering. Let $U$ be a local coordinate of $F$, and suppose that $Gf(U)$ and $4S^2$-bundle intersect only at $(X,Y)$. When we let $U_+$ denote a component $p^{-1}(U)$ in $\hat{F}$, $p^{-1}(U)$ consists of $U_+$ and $-U_+$. We regard each of them as a coordinate of $F$ via $p$. Those local orientations are opposite to each other. By the previous paragraph, $Gf(-U_+) = (-1,-1)(Gf(U_+))$. Thus $Gf(F)$ and $4S^2$-bundle intersect at $(-X,-Y)$ and the index at the point does not change, because the one local situation in $G(2,TM)$ is homeomorphic to the other under $(-1,-1)$ and both orientations of the surfaces change. This may be the very reason why the index of a complex point does not depend on the local orientation from our view point.




Around an intersection point (X,Y)

Around (-1,-1)(X,Y)

Finally, We must show that the algebraic index sum of those intersection points is zero. Since $f\colon F \to (M, J)$ is a generic immersion, the composition $f \circ p\colon \hat{F} \to (M, J)$ is also a generic immersion. By applying the conclusion of the formula for an oriented surface, the algebraic index sum of the intersection Int( $G(f \circ p)$, $4S^2$-bundle) is zero. On the other hand, by the previous paragraph, the algebraic index sum of the intersection of $Gf(F)$ with local orientation and $4S^2$-bundle is equal to $\frac{1}{2}$Int( $G(f \circ p)$, $4S^2$-bundle), which is zero. We have the formula. □

おわりに　この研究を始めたきっかけは、佐伯修氏から送られた手紙でした。その手紙の中で私の結果 [Y1] を曲面の複素多様体への曲面のはめ込みの問題に利用するアイデアが指摘されていました。その後、石川, 大本両氏から論文 [IO] を頂き、そこでは扱われなかった向き付け不能曲面の場合の話として今回の講演に至ります。３人の方々に感謝致します。ありがとうございました。

## References


[A] Audin, *Quelques remarques sur les surfaces lagrangiennes*, J.Geom.Phys. **7** (1990), 583–598.

[BF] T.Banchoff and F.Farris, *Tangential and Normal Euler Numbers, Complex points, and Singularities of Projections for Oriented Surface*, Pacific J. Math. **161 no.1** (1993), 1–24.

[Bi] E.Bishop, *Differentiable Manifolds in Complex Euclidean Space*, Duke. Math. J. **32** (1965), 1–21.

[CS] S.S.Chern and E.H.Spanier, *A Theorem on Orientable Surfaces in Four Dimensional Spaces*, Comm. Math. Helv. **25** (1951), 205–209.

[Fo] F.Forstneric, *Complex tangents of real surfaces in complex surfaces*, Duke Math. **67 no.2** (1992), 353–376.

[IO] G.Ishikawa and T.Ohmoto, *Local Invariants of Singular Surfaces in an Almost Complex Four Manifold*, Ann. of Global Analysis and Geometry **11** (1993), 125–133.

[L] Li,Bang He, *Generalization of Whitney-Mahowald Theorem*, preprint(1993).

[SS] O.Saeki and K.Sakuma, *Immersed Manifolds in Euclidean Spaces and Their Projections*, .

[W1] S.M.Webster, *Minimal surfaces in a Kähler surface*, J. Diff. Geo. **20** (1984), 463–470.

[W2] ———, *The Euler and Pontrjagin and Numbers of an n-Manifold in $C^n$*, Comm. Math. Helv. **60** (1985), 193–216.

[Y1] Y.Yamada, *An Extension of Whiyney's Formula*, Osaka J. Math. **32** (1995), 185–192.

[Y2] ———, *On the number of complex points of a surface in an almost complex 4-manifold*, preprint (1995).


Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba Meguro-ku
Tokyo, 153, Japan.