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The second pluri-genus of surface singularities

筑波大学数学系　渡辺公夫 (Kimio Watanabe)
筑波大学数学研究科　奥間智弘 (Tomohiro Okuma)

1 Preliminary

Let (X,x) be a normal n-dimensional isolated singularity over $\mathbf{C}$ and $f:(M,E)\rightarrow(X,x)$ a resolution of the singularity (X,x) with exceptional locus $E=f^{-1}(x)$. We say a resolution f is *good* if E is a divisor of normal crossings. Then the m-th *L^2-plurigenus* of (X,x) is an integer $\delta_m(X,x)$ which was introduced in [5] and can be computed as

$$\delta_m(X,x)=\dim_{\mathbf{C}}H^0(M-E,\mathcal{O}_M(mK))/H^0(M,\mathcal{O}_M(mK+(m-1)E)),$$

where K denotes the canonical divisor on M.

The invariant $p_g(X,x)$ called the *geometric genus* of a singularity (X,x) is defined by

$$p_g(X,x)=\dim_{\mathbf{C}}(R^{n-1}f_*\mathcal{O}_M)_x.$$

We note that $p_g(X,x)=\delta_1(X,x)$.

Normal surface singularities are classified as the following theorem.

(1.1) THEOREEM[1,Theorem 5.4]. *For a nomal surface singularity* (X, x), *we have the following*:

(1) $\delta_m(X, x) = 0$ *for all* $m \in \mathrm{N}$, *when* (X, x) *is a quotient singularity*;

(2) $\delta_m(X, x) \leq 1$ *for all* $m \in \mathrm{N}$ *and* $\delta_n(X, x) = 1$ *for some* $n \in \mathrm{N}$, *in one of the following cases*;

(a) (X, x) *is a simple elliptic singularity*;

(b) (X, x) *is a cusp singularity*;

(c) (X, x) *is the quotient by a finite group of a singularity of type* (a) *or* (b);

(3) $0 < \limsup_{m \to \infty} \delta_m(X, x)/m^2 < \infty$, *in other cases.*

(1.2) DEFINITION. A normal surface singularity (X, x) is *of general type* if which is in the class (3) above.

(1.3) DEFINITION. A normal surface singularity (X, x) is *minimally elliptic* if $p_g(X, x) = 1$ and (X, x) is a Gorenstein singularity.

2 The minimally elliptic singularities

Throughout this section, we assume that (X, x) is a minimally elliptic singularity, $f : (M, E) \to (X, x)$ the minimal good resolution with the irreducible decomposition $E = \bigcup_{i=1}^{k} E_i$ and K the canonical divisor on M which is supported on E.

Then $-K \geq E$, and $-K = E$ if and only if (X, x) is a simple elliptic or a cusp singularity. If (X, x) is not a simple elliptic singularity, then $E_i \cong \mathbf{P}^1$ for all i.

(2.1) DEFINITION. Let Z be the fundamental cycle on M. We say the dual graph of E is *obtained* from another singularity (X', x') if the self-intersection number of the fundamental cycle of (X', x') is -1 and the weighted dual graphs of (X, x) and (X', x') are same except for self-intersection numbers of the components of E with multiplicity 1 in Z.

In [6], we have followings.

(2.2) THEOREM. *For a minimally elliptic singularity (X, x),*

$$\delta_2(X, x) = \dim_{\mathbf{C}} H^1_E(M, \mathcal{O}_M(2K + E)).$$

If (X, x) is of general type, $\delta_2(X, x) = KD + 2$, where $D = -K - E$.

(2.3) COROLLARY. *If (X, x) is a hypersurface (resp. complete intersection), then $\delta_2(X, x) \leq 4$ (resp. 5).*

(2.4) PROPOSITION. *Let (X, x) be a minimally elliptic singularity of general type. Then $\delta_2(X, x) = 1$(resp. 2) if and only if (X, x) is* obtained *from a unimodal(resp. bimodal) singularity, and f is good if and only if*

$\delta_2(X,x) \geq 2$.

(2.5) QUESTION. *Does the inequality $\delta_2 \geq m(X,x)$ ($m(X,x)$ denotes the modality of a hypersurface singularity) hold ?*

That holds for quasi-homogeneous hypersurface singularities (see [7]).

3 The equisingular deformations

We follow the notation and terminology of the second section. We always assume that (X,x) is a minimally elliptic singularity.

Let ES be the equisingular deformation functor in the sense of [3]. By [2], an equisingular deformation of M induces a topologically constant deformation of a singularity (X,x).

By [3], ES is smooth and the tangent space of ES is $H^1(M,S)$, where $S = \mathcal{H}om_{\mathcal{O}_M}(\Omega^1_M(logE), \mathcal{O}_M)$ (which is a locally free sheaf of rank 2).

(3.1) DEFINITION. We define an invariant $q(X,x)$ callled *irregularity* by

$$q(X,x) = \dim_{\mathbf{C}} H^0(M-E, \Omega^1_{M-E})/H^0(M, \Omega^1_M).$$

If (X,x) is a simple elliptic or not a quasi-homogeneous singularity, then $q(X,x) = 0$, and $q(X,x) = 1$ for every other(cf. [4, THEOREM 1.9]).

We denote by $h^i(\cdot)$(resp. $h^1_E(\cdot)$) the dimension of C-vector space $H^i(\cdot)$(resp. $H^1_E(\cdot)$).

(3.2) PROPOSITION. *If (X, x) is of general type, then*

$$h^1(S) = q(X,x) + h^0(D, \Omega^1_M(log E) \otimes \mathcal{O}_D(-E)),$$

where $D = -K - E$.

Proof. By the duality, $h^1(S) = h^1_E(\Omega^1_M(log E) \otimes \mathcal{O}(K))$. By Wahl's vanishing theorem, $h^1(\Omega^1_M(log E) \otimes \mathcal{O}(K)) = 0$. Hence we have

$$h^1_E(\Omega^1_M(log E) \otimes \mathcal{O}(K)) = \dim_{\mathbf{C}} H^0(M-E, \Omega^1_{M-E})/H^0(\Omega^1_M(log E) \otimes \mathcal{O}(K))$$

and

$$h^0(D, \Omega^1_M(log E) \otimes \mathcal{O}_D(-E)) = \dim_{\mathbf{C}} H^0(\Omega^1_M(log E) \otimes \mathcal{O}(-E))/H^0(\Omega^1_M(log E) \otimes \mathcal{O}(K)).$$

From the exact sequence

$$0 \to H^0(\Omega^1_M(log E) \otimes \mathcal{O}(-E)) \to H^0(\Omega^1_M) \to H^0(\oplus \Omega^1_{E_i}) = 0,$$

and inclusions

$$H^0(\Omega^1_M(log E) \otimes \mathcal{O}(K)) \subseteq H^0(\Omega^1_M(log E) \otimes \mathcal{O}(-E)) \subseteq H^0(\Omega^1_M) \subseteq H^0(M-E, \Omega^1_M),$$

we have the assertion of the proposition.

In a similar way as above, we have $h^1(S) = 1$ for simple elliptic singularities and $h^1(S) = 0$ for cusp singularities.

(3.3) EXAMPLE. Let (X,x) be of general type and $E = \bigcup_{i=0}^{r} E_i$ the irreducble decomposition. Assume that dual graph of E is star-shaped such that $E_0E_i = 1$ for $i = 1, \dots, r$, $E_iE_j = 0$ if $1 \leq i < j \leq r$ and $E_0^2 = -r + 2$.

Then $K = -2E_0 - \sum_{i=1}^{r} E_i$, and $D = -K - E = E_0$. Hence $\delta_2(X,x) = KD + 2 = r - 2$.

There is an isomorphism

$$\Omega_M^1(logE) \otimes \mathcal{O}_{E_0}(-E) \cong \mathcal{O}_{E_0}(-4+r) \oplus \mathcal{O}_{E_0}(-2).$$

By (3.2), $h^1(S) = q(X,x) + r - 3$. Hence $\delta_2(X,x) = h^1(S) - q(X,x) + 1$.

If (X,x) is a quasi-homogeneous hypersurface singularity, and moreover if the invariance of Milnor's number implies the invariance of the topological type, then $\delta_2(X,x) = m(X,x)$.

By [4, (1.5),(1.6)], the exterior differentiation gives an exact sequence

$$0 \to d\mathcal{O}(K) \to \Omega_M^1(logE) \otimes \mathcal{O}(K) \to \Omega_M^2 \otimes \mathcal{O}(K+E) \to 0$$

and isomorphisms $H^i(\mathcal{O}(K)) \cong H^i(d\mathcal{O}(K))$ for all i, where we consider $\mathcal{O}(K)$ an ideal sheaf of $\mathcal{O}_M$.

Therer is an exact sequence

$$0 \to H^0(d\mathcal{O}(K)) \to H^0(M-E, d\mathcal{O}) \to H_E^1(d\mathcal{O}(K))$$

$$\to H^1(d\mathcal{O}(K)) \cong H^1(\mathcal{O}(K)) = 0.$$

If (X,x) is of general type, then $H_{\{x\}}^1(d\mathcal{O}_X) = 0$ by [4,(1.13.4)]. Hence

$H^0(X, d\mathcal{O}_X) \cong H^0(M - E, d\mathcal{O})$, and the map $H^0(d\mathcal{O}(K)) \to H^0(M - E, d\mathcal{O})$ is surjective. Hence $H^1_E(d\mathcal{O}(K)) = 0$.

By the exact sequence

$$H^1_E(d\mathcal{O}(K)) \to H^1_E(\Omega^1_M(logE) \otimes \mathcal{O}(K)) \to H^1_E(\mathcal{O}(2K + E))$$

$$\to H^2_E(d\mathcal{O}(K)) \to H^2_E(\Omega^1_M(logE) \otimes \mathcal{O}(K)),$$

we have $\delta_2(X, x) = h^1(S) + \alpha$ for the singularity (X, x) of general type, where

$$\alpha = \dim_{\mathbf{C}}\mathrm{Ker}(H^2_E(d\mathcal{O}(K)) \to H^2_E(\Omega^1_M(logE) \otimes \mathcal{O}(K))).$$

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