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SEMIALGEBRAIC VERSION OF THOM'S SECOND ISOTOPY LEMMA

MASAHIRO SHIOTA

Dept. of Math., Nagoya University

In real singularities the most important maps are polynomial ones. Moreover, even if a specialist states a theorem by C^∞ maps, he actually consider polynomial maps in mind. So it is natural to restrict our interest to polynomial maps. There are two kinds of equivalence relations on polynomial maps: C^∞ equivalence and C^0 equivalence. Let us consider C^0 equivalence. It is said that C^0 equivalence is visual. But this is not correct, and means only that we consider problems without worrying about differentiability. C^0 equivalence is artificial and unnatural. By unnaturalness there are many strange phenomena. For example, recall the King's example of polynomial function germs $f, g\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ with isolated singularities such that $(\mathbf{R}^n, f^{-1}(0))$ and $(\mathbf{R}^n, g^{-1}(0))$ are C^0 equivalent but f and g are not $R-L$ C^0 equivalent [K]. The homeomorphism germ of C^0 equivalence is constructed by infinite process, and since the process cannot be finitely controlled we can not extend the equivalence to $R-L$ C^0 equivalence of f and g. The example is a counter-example to a Thom's conjecture. We can not expect a beautiful theory on C^0 equivalence.

I propose semialgebraic equivalence in place of C^0 equivalence, which is defined by a homeomorphism with semialgebraic graph. Semialgebraic equivalence is strictly stronger than C^0 equivalence. Namely,

(1) there exist two polynomial function germs which are C^0 equivalent but not semialgebraically equivalent [S].

On the other hand, semialgebraic equivalence is weaker than C^1 equivalence. Indeed,

(2) two polynomial function germs are semialgebraically equivalent if they are C^1 equivalent [S].

A good property is the following, which is a positive answer to the above Thom's conjecture.

(3) For two polynomial function germs $f, g\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, if $(\mathbf{R}^n, f^{-1}(0))$ and $(\mathbf{R}^n, g^{-1}(0))$ are semialgebraically equivalent, f and g are semialgebraically equivalent up to $\pm$, namely, $|f|$ and $|g|$ are semialgebraically equivalent [S].

Behavior of semialgebraic functions at infinity is strongly restricted. This is a reason why I expect a good theory of semialgebraic equivalence. Here we note only that

(4) there exist two polynomial functions on $\mathbf{R}^8$ which are C^ω equivalent but not semialgebraically equivalent [S].

Almost all the known positive results on C^0 equivalence were proved only by the Thom's second isotopy lemma. Hence the first step to construct a theory of semialgebraic equivalence is to prove its semialgebraic version.

Theorem [S]. *Let $\{X_i\}$ and $\{Y_j\}$ be semialgebraic C^1 Whitney stratifications of closed semialgebraic sets X and Y, respectively, in $\mathbf{R}^n$, and let $f: X \to Y$ be a proper semialgebraic C^1 map such that for each i, $f(X_i)$ equals some Y_j and $f|_{X_i}$ is a C^1 submersion onto Y_j. Let $p: Y \to \mathbf{R}^m$ be a proper semialgebraic C^1 map such that for each j, $p|_{Y_j}$ is a C^1 submersion onto $\mathbf{R}^m$. Assume f is sans éclatement. Set*

$$X(0) = (p \circ f)^{-1}(0), \quad Y(0) = p^{-1}(0).$$

There exist semialgebraic C^0 maps $\rho: X \to X(0)$ and $\xi: Y \to Y(0)$ such that $(\rho, p \circ f): X \to X(0) \times \mathbf{R}^m$ and $(\xi, p): Y \to Y(0) \times \mathbf{R}^m$ are homeomorphisms and the diagram

$$\begin{array}{ccc} X & \xrightarrow{(\rho, p\circ f)} & X(0) \times \mathbf{R}^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{id}} \\ Y & \xrightarrow[(\xi, p)]{} & Y(0) \times \mathbf{R}^m \end{array}$$

is commutative.

One of the corollaries is a version of Mather's C^0 Stability Theorem.

Corollary. *Let $M \subset \mathbf{R}^n$ be a compact nonsingular algebraic variety. The family of semialgebraically stable polynomial maps is dense in the polynomial maps from M to $\mathbf{R}^m$.*

Let r be a large integer and let M_1 and M_2 be semialgebraic C^r manifolds in $\mathbf{R}^n$. The family of semialgebraically stable semialgebraic C^r maps is dense in the semialgebraic C^r maps from M_1 to M_2. (See [S] for the topology.)

References

[K] H. C. King, *Real analytic germs and their varieties at isolated singularities*, Inv. Math. **37** (1976), 193–199.

[S] M. Shiota, *Geometry of subanalytic and semialgebraic sets* (to appear).