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SEQUENTIAL ORDER OF PRODUCTS OF FRÉCHET TOPOLOGIES

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ABSTRACT. The sequential order of a topological space is the least ordinal for which the corresponding iteration of the sequential closure is idempotent. Lower estimates for the sequential order of the product of two regular Fréchet topologies and upper estimates for the sequential order of the product of two subtransverse topologies are given in terms of their fascicularity and sagittality. Consequently it is possible for every countable ordinal α to construct two Fréchet topologies with the sequential order of their product equal to α. This paper is a short version of [6].

The sequential order $\sigma(x)$ of a point x of a topological space X is the least ordinal α such that whenever x belongs to an iterated sequential closure of a set, then it belongs to its α-iterated sequential closure. The sequential order of X is equal to $\sup_{x\in X}\sigma(x)$.

Sequential order is always less than or equal to ω_1. Recall that a topology is sequential if each sequentially closed set is closed. Fréchet topologies are precisely the sequential topologies of sequential order less than or equal to 1. It is well-known [1, 10, 8, 13] that the product of two Fréchet topologies needs neither be sequential nor of order less than or equal to 1. This paper is devoted to the study of the sequential order of products of Fréchet topologies.

In [14] T. Nogura and A. Shibakov investigate the sequential order of products of sequential topologies under the requirement that the products be also sequential. They prove in particular that if the product of two Fréchet topologies admitting point countable k-networks ([1]) is sequential, then its sequential order is less than or equal to 2. On the other hand they construct [14, Example 2.13] two Fréchet topologies

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[1] A family $\mathcal{A}$ of sets, closed under finite unions is a k-network if for each compact set K and each open O with $K \subset O$, there is $A \in \mathcal{A}$ such that $K \subset A \subset O$.

with pointwise countable k-networks such that the sequential order (in our sense) of their product equals 3.

Here we show that for every countable ordinal α, there exists a Lašnev space ([2]) whose square is of sequential order α (as Lašnev spaces are Fréchet spaces with point countable k-networks, our square is not sequential for $\alpha > 2$, because of the above mentioned result).

In a forthcoming paper [15] T. Nogura and A. Shibakov construct, under **CH** for each $\alpha \leq \omega_1$, two strongly Fréchet topologies the product of which is sequential and of sequential order α.

Our estimates for the sequential order of products are formulated in terms of two ordinals associated with every point x of a topological space: fascicularity $\lambda(x)$ and sagittality $\mu(x)$. The first corresponds to the rank of multifans converging to x, the second to the rank of arrows (i.e sequences of multifans) converging to x. If X, Y are regular Fréchet topological spaces, then the sequential order $\sigma(x, y)$ is not less than

$$1 + \left[\left(\lambda(x) \wedge \mu(y)\right) \vee \left(\mu(x) \wedge \lambda(y)\right)\right]$$

for every $x \in X$ and $y \in Y$. The above quantity is an upper bound for the sequential order $\sigma(x, y)$ provided that X and Y are sequential and subtransverse (we say that a topological space X is subtransverse if for every injective sequence (x_n) converging to x, there exists a subsequence (n_k) and a sequence Q_k with $Q_k \in \mathcal{N}(x_{n_k})$ such that for each neighborhood Q of x, there is k_Q for which $Q_k \subset Q$ as $k \geq k_Q$). Lašnev spaces are Fréchet subtransverse and normal, so that in case of Lašnev spaces the above quantity is equal to the sequential order $\sigma(x, y)$.

Our method hinges on the following general characterization: if α is the least ordinal such that x belongs to the α-iteration of the sequential closure of a set A, then there exists a multisequence of rank α on A which converges to x.

All the topologies considered throughout this paper are Hausdorff.

1. Sequential order and admissible multisequences

We denote by $cl_{seq}A$ the *sequential closure* of A, i.e, the union of the limits of all convergent sequences valued in A. One defines $cl_{seq}^0 A = A$ and for each ordinal $\alpha > 0$,

$$cl_{seq}^{\alpha} A = cl_{seq} \bigcup_{\beta<\alpha} cl_{seq}^{\beta} A. \ (^3)$$

The least ordinal α for which cl_{seq}^{α} is idempotent is called the *sequential order* of the topological space and is denoted by $\sigma(X)$.

[2] i.e., closed image of metrizable spaces.

[3] Some authors e.g., Nogura and Shibakov [14], define limit powers by $cl_{seq}^{\alpha} A = \bigcup_{\beta<\alpha} cl_{seq}^{\beta} A$.

If $x \in cl_{seq}^{\omega_1} A$, then the *sequential order* $\sigma(x; A)$ (of x with respect to A) is the least ordinal α such that $x \in cl_{seq}^{\alpha} A$. The sequential order $\sigma(x)$ is defined by

$$\sigma(x) = \sup\{\sigma(x; A) : A \subset X,\ x \in cl_{seq}^{\omega_1} A\}.$$

Consequently, $\sigma(X) = \sup_{x \in X} \sigma(x)$. Remark that for every x, one has $\sigma(x) \leq \omega_1$.

Consider the set $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ of finite sequences valued in $\mathbb{N}$ ordered by inclusion (denoted by $\sqsubseteq$). In what follows (t, s) denotes the concatenation of the finite sequences t and s. It follows that $r \sqsubseteq s$ whenever there exists t such that $s = (r, t)$.

Following D. Fremlin [9] we consider the subsets T of $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ that are *well-capped* trees (i.e., such that every non empty subset of T has a maximal element in T)[4] that fulfill

$$s \sqsubseteq t,\ t \in T \implies s \in T, \tag{1.1}$$

$$\underset{t \in T}{\forall} \big(\underset{n \in \mathbb{N}}{\exists} \ (t, n) \in T \implies \underset{n \in \mathbb{N}}{\forall} \ (t, n) \in T \big). \tag{1.2}$$

From now on we understand by a *tree* a well-capped tree in $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ fulfilling (1.1), (1.2). The elements of a tree are called *indices.* By a *subtree* we understand a subset of a tree which is a tree in the above sense.

Denote by $l(t)$ the length of the finite sequence t. Every well-capped tree T admits the unique rank function:

$$r(t) = r(t; T) = \min\{\alpha \in Ord : \underset{s \sqsupset t}{\forall} r(s) < \alpha\}.$$

In our case (of well-capped trees) one has

$$\begin{aligned} t \in \max T &\implies r(t) = 0 \\ t \notin \max T &\implies r(t) = \sup_{n \in \mathbb{N}} (r(t, n) + 1). \end{aligned} \tag{1.3}$$

For each $\alpha < \omega_1$, there exists a tree T of rank α (i.e., $r(\varnothing; T) = \alpha$) [9].

Let T be a tree. We define on T the *irreducible convergence*: $\lim_k t_k = t$ if and only if that for almost all k, either $t_k = t$ or $t_k = (t, n_k)$ with $\lim_k n_k = \infty$. This convergence is Urysohn [5]. The associated topology

[4] In other words, T considered with the inverse order is well-founded.

[5] A sequence convergence is *Urysohn* if $\lim_n x_n = x$ and $\lim_k n_k = \infty$ imply $\lim_k x_{n_k} = x$ and, if a sequence does not converge to x, then there exists a subsequence such that none of its subsequences converges to x.

[6], i. e., the finest topology coarser than the irreducible convergence is called the *irreducible topology.*

Of course, if a tree T is *monotone* [7], i.e., has the property that for every $t \notin \max T$, the sequence $r(t,n)$ is increasing, then

$$r(t) = \lim_n (r(t,n) + 1). \tag{1.4}$$

An Urysohn convergence on T is said to be *admissible* if it is coarser than the irreducible convergence and if for every $t \in T \setminus \max T$, one has $\lim_k t_k = t$ implies that $t_k \sqsupseteq t$ and if moreover (t_k) is such that $t_k \sqsupseteq (t, n_k)$, then $\lim_k n_k = \infty$ and

$$\liminf_k (r(t_k) + 1) = r(t). \tag{1.5}$$

The topology associated with an admissible convergence is called *admissible.* Of course, if $r(\varnothing) < \omega_0$, then the only admissible convergence is that irreducible.

Because of (1.5) and (1.3), for every t in a monotone tree T equipped with the irreducible topology,

$$\sigma(t; \max T) = r(t; T). \tag{1.6}$$

Let T be a tree. A *multisequence* in X is a mapping $f : \max T \longrightarrow X$; the *extension* is a mapping $\tilde{f} : T \longrightarrow X$ such that $\tilde{f}(t) = f(t)$ for every $t \in \max T$. We shall use the term *multisequence* also for such extensions [8]. The *rank* $r(f)$ of a multisequence f is, by definition, the rank of the underlying tree. The *initial restriction* of a multisequence $f : T \to X$ is the restriction of f to a subtree S of T.

An (extended) multisequence $g : S \longrightarrow X$ is a *transmultisequence* of $f : T \longrightarrow X$ if there exists a mapping $h : S \longrightarrow T$ such that $g = f \circ h$ and

$$h(\varnothing) = \varnothing, \tag{1.7}$$

$$\forall_{s \in S} \quad h(s,n) \sqsupseteq (h(s), m_n) \text{ with } \lim_n m_n = \infty, \tag{1.8}$$

$$h(\max S) \subset \max T. \tag{1.9}$$

A *submultisequence* of $f : T \longrightarrow X$ is a transmultisequence such that

$$\forall_{s \in S} \quad h(s,n) = (h(s), m_n) \text{ with } \lim_n m_n = \infty. \tag{1.10}$$

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[6]The associated topologies of Urysohn convergences with the unicity of limits are sequential [11].

[7]Each tree includes a monotone tree of the same rank.

[8]We are grateful to Professor A. Kato for having drawn our attention to [2, 12] were (extended) multisequences with some extra topological properties were introduced.

An (extended) multisequence $f : T \longrightarrow X$, valued in a topological space X, *converges* to a point x if for every $t \in T\setminus \max T$, $\lim_n f(t,n) = f(t)$ and $x = f(\varnothing)$. The *sequential order* of a convergent multisequence f is defined by $\sigma(f) = \sigma(f(\varnothing); f(\max T))$. The sequential order $\sigma(f)$ is always less than or equal to the rank $r(f)$.

An injective convergent multisequence $f : T \to X$ is said to be *irreducible (resp. admissible)* if the initial convergence on T with respect to f is irreducible (resp., admissible).

One might suspect that if $\sigma(x, A) = \alpha$, then there exists an irreducible multisequence $f : \max T \longrightarrow A$ that converges to x and such that $r(f) = \alpha$. This is in general not the case. There exists a topological space and a point therein of sequential order ω_0 with no irreducible multisequence converging to it.

Theorem 1.1. *If $\sigma(x, A) = \alpha$, then there exists a monotone admissible multisequence $f : \max T \longrightarrow A$ that converges to x and such that $r(f) = \alpha$.*

Corollary 1.2. *If $\sigma(x; A) < \omega_0$, then there exists an irreducible multisequence f on A converging to x and such that $\sigma(f) = r(f) = \sigma(x; A)$.*

2. Multifans and arrows

A convergent multisequence $f : T \to X$ is called a *multifan* if for each t of even length in $T \setminus \max T$, one has $f(t,n) = f(t)$ for each $n \in \mathbb{N}$. A convergent multisequence $f : T \to X$ is said to be an *arrow* if for every t in $T \setminus \max T$ of odd length, one has $f(t,n) = f(t)$ for each $n \in \mathbb{N}$. In other words, f is an arrow if for each n, the restriction of f to $T_n := \{s : (n,s) \in T\}$ is a multifan. A multifan (resp. arrow) $f : T \to X$ is *injective* if it is injective modulo the equivalence relation: if t is of even (resp. odd) length in $T \setminus \max T$, then $t \equiv (t,m)$ for every $m \in \mathbb{N}$. Let $f : T \to X$ be a multifan and R the subtree of T obtained by removing all maximal indices of odd length. If $f : R \to X$ is injective, then we define its *fascicularity* $\lambda(f)$ as the rank $r(\varnothing; R)$.

Similarly, if $f : T \to X$ is an arrow and if f restricted to the subtree R of T obtained by removing all maximal indices of even length is injective, then we define its *sagittality* $\mu(f)$ as the rank $r(\varnothing; R)$. If $R = \varnothing$, then we convene that $\mu(f) = -1$.

Consequently, if f is a multifan and if g is an arrow, then

$$\lambda(f) \le r(f) \le 1 + \lambda(f), \qquad \mu(g) \le r(g) \le 1 + \mu(g). \tag{2.1}$$

If f is a monotone multifan (i. e., if the corresponding tree is monotone) and f_n is its n-th arrow, then $\lambda(f) = \lim_n(\mu(f_n)+1)$; if g is a monotone arrow and g_n is its n-th multifan, then $\mu(f) = \lim_n(\lambda(f_n) + 1)$.

An injective multifan $f : T \to X$ is *untraversable* if for every t of even length, and each $t_k \sqsupset (t, n_k)$ such that $\lim_k f(t_k) = f(t)$, the sequence (n_k) is bounded. In paticular, a fan is untraversable if no sequence $(x_{(n_p,k_p)})_p$ with n_p tending to ∞ converges to x. Untraversable fans are frequently denoted by S_ω.

The *fascicularity* $\lambda(x)$ of a point x is the least upper bound of $\lambda(f)$ of all the untraversable multifans f converging to x. The *sagittality* $\mu(x)$ is the least upper bound of $\mu(g)$ of all the untraversable arrows g converging to x. These bounds do not change if we consider only the monotone untraversable multifans and arrows. Therefore and because every untraversable multifan is composed of untraversable arrows, $\mu(x) + 1 \geq \lambda(x)$.

3. Bounds for sequential order of products

We say that a multisequence $f : T \to X$ is *transversally closed* if for each $t \in T \setminus \max T$, $t_k \sqsupseteq (t, n_k)$ such that $\lim_k f(t_k) = x$ and $\lim_k n_k = \infty$ implies that $x = f(t)$.

It follows from [16, Theorem 3.8] of T. Nogura and Y. Tanaka that for each untraversable fan $(x_{(n,k)})$ converging to x in a regular Fréchet space, there exists a mapping $h : \mathbb{N} \to \mathbb{N}$ such that $\{x\} \cup \{x_{(n,k)} : k \geq h(n),\ n \in \mathbb{N}\}$ is closed. For fans closedness and transversal closedness coincide. Although the following theorem assures only the transversal closedness of a submultifan, the submultifan constructed in the proof is such that the proof extends the above quoted theorem of Nogura and Tanaka.

Theorem 3.1. *Each untraversable multifan in a regular Fréchet space includes a transversally closed submultifan.*

Theorem 3.2. *If X and Y are regular Fréchet spaces, then*

$$\sigma(x, y) \geq 1 + \big[\big(\lambda(x) \wedge \mu(y)\big) \vee \big(\mu(x) \wedge \lambda(y)\big)\big]. \tag{3.1}$$

An upper bound for the sequential order of products is given in the case of subtransverse topologies that we define below. A topology is *transverse* if for every injective sequence (x_n) converging to x, there exists a sequence Q_n with $Q_n \in \mathcal{N}(x_n)$ such that

$$\lim_n Q_n = x,$$

i.e., for each $Q \in \mathcal{N}(x)$ there exists $n_Q \in \mathbb{N}$ with $Q_n \subset Q$ for $n \geq n_Q$. A topology is *subtransverse* if for every injective sequence (x_n) converging to x, there exists a subsequence (n_k) and a sequence Q_k

with $Q_k \in \mathcal{N}(x_{n_k})$ such that $\lim_k Q_k = x$. A convergent bisequence

$$x_{(n,k)} \xrightarrow[k]{} x_n \xrightarrow[n]{} x, \tag{3.2}$$

with $\lim_n x_{(n,k_n)} = x$ for every sequence (k_n) is called *transverse.* One observes that the topology induced on such a bisequence is first-countable. A topology is *sequentially transverse* if for every convergent injective bisequence there exists $f : \mathbb{N} \to \mathbb{N}$ such that the bisequence restricted to $x_{(n,k)}$ such that $k \geq f(n)$ for all n is transverse. A topology is *sequentially subtransverse* if every convergent bisequence admits a tranverse subbisequence.

In [17] Popov and Rančin say that a topological space X is a Φ-*space* if for every $A \subset X$ and for each $x \in \operatorname{cl} A$, there exists a sequence (Q_n) of open sets such that $\lim_n Q_n = x$ and $Q_n \cap A \neq \varnothing$ for each n. In [4, Proposition 7] it is shown that a topological space is a Φ-space if and only if it is a sequential subtransverse space.

Each sequential sequentially subtransverse space is Fréchet.

In [18] P. Simon constructed a compact Fréchet topology whose square is not Fréchet. The Simon topology is an example of a Fréchet non sequentially subtransverse space.

On the other hand, each Fréchet space with a point-countable k-network is sequentially transverse. This fact follows from [14, Lemma 2.6] where T. Nogura and A. Shibakov prove (more than they announce) that in each Fréchet space with a point-countable k-network for every convergent bisequence (3.2), there exists $h : \mathbb{N} \to \mathbb{N}$ such that $\{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{(n,k)} : k \geq h(n), n \in \mathbb{N}\}$ is compact and the points of the form $x_{(n,k)}$ are isolated in it.

There exist Fréchet transverse topologies without point-countable k-network. It is shown in a forthcoming paper [4, Example 10] that the Σ-product of uncountably many copies of the discrete two-point space is Fréchet sequentially subtransverse not subtransverse space. [4, Example ?] shows the existence of subtransverse non transverse spaces under the provision of Martin's Axiom.

Recall that a closed continuous image of a metrizable space is called a *Lašnev space.* It is known that every Lašnev space is a Fréchet space with a point countable k-network [7]. In [17] Popov and Rančin show that each Lašnev space is subtransverse. Unaware of their result, we have proved in [6]that Lašnev spaces are transverse.

There exists a transverse topology without point countable k-network.

Theorem 3.3. *If X and Y are sequential transverse spaces, then*

$$\sigma(x,y) \leq 1 + [(\lambda(x) \wedge \mu(y)) \vee (\mu(x) \wedge \lambda(y))]. \tag{3.3}$$

As already mentioned, Lašnev spaces are normal Fréchet and transverse. Hence by Theorems 3.2 and 3.3, we have

Theorem 3.4. *If X and Y are Lašnev spaces, then*

$$\sigma(x,y) = 1 + \left[\left(\lambda(x) \wedge \mu(y)\right) \vee \left(\mu(x) \wedge \lambda(y)\right)\right]. \tag{3.4}$$

On the other hand,

Theorem 3.5. *For every ordinal $\alpha \le \omega_1$, there exists a Lašnev space such that the sequential order of its square is α.*

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