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SOME REMARKS ON THE DUGUNDJI EXTENSION THEOREMS

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1. Results that are known or easily proved

Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Let $C(X, Z)$ be the linear space of all continuous mappings from X to Z. A linear transformation $u: C(A, Z) \rightarrow C(X, Z)$ is said to be a *Dugundji extender* if u satisfies the following conditions: For each $f \in C(A, Z)$,
(a) $u(f)$ is an extension of f, and
(b) the range of $u(f)$ is contained in the closed convex hull of the range of f.

The study of this area is initiated by Dugundji [2]. He proved that for every closed subspace A of a metrizable space X there exists a Dugundji extender $u: C(A, \mathbb{R}) \rightarrow C(X, \mathbb{R})$. Michael ([8]) noticed that the Dugundji extender constructed by Dugundji is continous with respect to the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

We shall consider the Dugundji extention theorems on product spaces.

Definition 1.1. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is *D(Z)-embedded* in X if there is a Dugundji extender $u: C(A, Z) \rightarrow C(X, Z)$. Furthermore, we say that A is *D-embedded* in X if A is D(Z)-embedded in X for every locally convex linear topological space Z.

Definition 1.2. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is $\pi_{D(Z)}$*-embedded* in X if for every space Y there is a Dugundji extender $u: C(A \times Y, Z) \rightarrow C(X \times Y, Z)$. Furthermore, A is said to be π_D*-embedded* in X if A is $\pi_{D(Z)}$-embedded in X for every locally convex linear topological space Z.

Definition 1.3. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is *continuously* $\pi_{D(Z)}$*-embedded (resp.* π_D*-embedded)* in X if we can choose the Dugundji extender u as is continuous with respect the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

For a space X and a locally convex linear topological space Z we denote $C_u(X, Z)$ the linear topological space of all continuous mappings from X to Z with the uniform convergence topology, i.e., the sets of the form $V(f) = \{g \in C(X, Z) : g(x) - f(x) \in V\}$, where V is a neighborhood of the origin of Z consists a basic neighborhoods of $f \in C_u(X, Z)$. Let $C_{co}(X, Z)$ be the linear topological space of all continuous mappings from X to Z with the compact-open topology.

A mapping $f : X \to Y$ is called a *Z-map* if $f(Z)$ is closed for every zero-set Z of X. Then we have the following.

Theorem 1.1. *Let X and Y be spaces and A a D-embeded subspace of X. Let $p_A : A \times Y \to A$ and $p_Y : A \times Y \to Y$ be the projections. If either of the following conditions is satisfied, then $A \times Y$ is D-embedded in $X \times Y$:*

(1) p_A is a Z-map.

(2) p_Y is a Z-map and there is a continuous Dugundji extender $u : C_u(A, Z) \to C_u(X, Z)$ for every locally convex linear topological space Z.

Theorem 1.2. *([4]) Let X and Y be spaces, A a closed subspace of X and Z a locally convex linear topological space. Suppose that X is locally compact or $X \times Y$ is a k-space. If there exists a continous Dugundji extender $u : C_{co}(A, Z) \to C_{co}(X, Z)$, then $A \times Y$ is D(Z)-embedded in $X \times Y$.*

Remark. In Theorem 1.2, the continuoity of the Dugundji extender u can not be dropped. In fact, let $X = [0, \omega_1] \times [0, \omega] - \{(\omega_1, \omega)\}$ and $A = [0, \omega_1) \times \{\omega\}$ be the closed subspace of X. It is clear that A is D($\mathbb{R}$)-embedded in X. Let $Y = [0, \omega_1]$ be the space with the following topology: For each $y < \omega_1$ y is an isolated point of Y and ω_1 has a neighborhood base of the usual order topology. It follows that $A \times Y$ is not C-embedded in $X \times Y$, and hence $A \times Y$ is not D($\mathbb{R}$)-embedded in $X \times Y$.

In [9] and [10], Stares proved that every closed subspace of spaces satisfying the decreasing (G) is π-embedded and every such space has the Dugundji extension property. Before stating the theorem, we recall the definition of spaces satisfying the decreasing (G) from [1]. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ be a collection of subsets of X, where $\mathcal{W}(x) = \{W(x, n) : n \in \omega\}$ such that $x \in W(x, n)$ for every $x \in X$ and $n \in \omega$. Then we say that $\mathcal{W}$ is decresing if $W(x, n+1) \subset W(x, n)$ for every $n \in \omega$, and $\mathcal{W}$ satisfies (G) if

(G) for each $x \in X$ and each open set U with $x \in U$ there is an open neighborhood $V = V(x, U)$ of x such that $y \in V$ implies $x \in W(y, s) \subset U$ for some $s \in \omega$.

We say that a space X satisfies the decreasing (G) if there is a collection $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying decreasing (G). We notice that every stratifiable space satisfies the decreasing (G) ([10]). Now, we have the following.

Theorem 1.3. *Let X be a regular space satisfying the decreasing (G) and A a closed subspace of X. Then A is continuously π_D-embedded in X.*

2. Results about GO-spaces

In [7], we proved that for a perfectly normal GO-space X with $E(X)$ is σ-discrete in X, a closed subspace A of X and Z a locally convex linear topological space Z, there is a Dugundji extender u from $C(A, Z)$ to $C(X, Z)$, where $E(X) = \{x \in X : (\leftarrow, x]$ or $[x, \to)$ is open in X $\}$. We extend the theorem above as follows.

Theorem 2.1. *Let X be a perfectly normal GO-space such that $E(X)$ is σ-discrete in X. Then every closed subspace A of X is continuously π_D-embedded in X.*

Proof. Let A be a closed subspace of X. Then $X - A$ is the union of a disjoint family $\mathcal{U}$ of convex components of $X - A$. Since X is perfectly normal, it follows from [3, Theorem 2.4.5] that $\mathcal{U}$ is σ-discrete in X. Let $\mathcal{U} = \cup_{n=1}^{\infty}\mathcal{U}_n$, where $\mathcal{U}_n$ is discrete in X. Similarly, let $\operatorname{Int} A = \bigcup \mathcal{V}$, where $\mathcal{V} = \cup_{n=1}^{\infty}\mathcal{V}_n$ is a disjoint and σ-discrete family of convex components of $\operatorname{Int} A$. For each $U \in \mathcal{U}$ we choose $x(U) \in U$. We put $M_{\mathcal{U}} = \{x(U) : U \in \mathcal{U}\}$. For each convex open set C in X, we put

- $l(C) = \max\{a \in A : a < x \text{ for all } x \in C\}$, and
- $r(C) = \min\{a \in A : a > x \text{ for all } x \in C\}$,

if the righthand of the above equations exist.

Then for each n, we put $\mathcal{U}_n^{\ell} = \{U \in \mathcal{U}_n : l(U) \text{ exists}\}$ and $\mathcal{U}_n^{r} = \{U \in \mathcal{U}_n : r(U) \text{ exists}\}$. Similarly, we define $\mathcal{V}_n^{\ell}$ and $\mathcal{V}_n^{r}$. Furtheremore, we put

- $L_n = \{l(U) : U \in \mathcal{U}_n^{\ell}\}$,
- $R_n = \{r(U) : U \in \mathcal{U}_n^{r}\}$,
- $L_n' = \{l(V) : V \in \mathcal{V}_n^{\ell}\}$, and
- $R_n' = \{r(V) : V \in \mathcal{V}_n^{r}\}$.

It is easy to see that all of L_n, R_n, L_n' and R_n' are closed discrete in X. Let $L = \cup_{n=1}^{\infty}L_n$, $R = \cup_{n=1}^{\infty}R_n$, $L' = \cup_{n=1}^{\infty}L_n'$ and $R' = \cup_{n=1}^{\infty}R_n'$. Furthermore, we put

$$B = \{a \in A - (L \cup R) : a \in \overline{\bigcup \mathcal{U}^-(a)}^X \cup \overline{\bigcup \mathcal{U}^+(a)}^X\},$$

where $\mathcal{U}^-(a) = \{U \in \mathcal{U} : x(U) < a\}$ and $\mathcal{U}^+(a) = \{U \in \mathcal{U} : x(U) > a\}$. Let

$$M = M_{\mathcal{U}} \cup L \cup R \cup L' \cup R' \cup (E(X) \cap A) \cup B.$$

Then M is a GO-space and $D = M - B$ is σ-discrete in M. Since $E(M) \subset D$ and D is dense in M, it follows from [3, Theorem 3.1] that M is metrizable. Then there exists a compatible metric ρ on M bounded by 1.

We shall define a mapping $\varphi : X \to 2^A$. Let $x \in X$. If $x \in A$, then we put $\varphi(x) = \{x\}$. Let $x \in X - A$. Then there is $U \in \mathcal{U}_n$ such that $x \in U$.

Case 1. Suppose that $U \in \mathcal{U}_n^{\ell} \cap \mathcal{U}_n^{r}$. If $U = \{x\}$, we put $\varphi(x) = \{\ell(U)\}$. If U contains at least two points, we choose points $s(U)$ and $t(U)$ of U such that $s(U) < t(U)$. We put

$$\varphi(x) = \begin{cases} \{\ell(U)\}, & \text{if } x < s(U), \\ \{\ell(U), r(U)\}, & \text{if } s(U) \le x \le t(U), \\ \{r(U)\}, & \text{if } x > t(U). \end{cases}$$

Case 2. If $U \in \mathcal{U}_n^{\ell}$ and $U \notin \mathcal{U}_n^{r}$, then we put $\varphi(x) = \{\ell(U)\}$.

Case 3. If $U \notin \mathcal{U}_n^{\ell}$ and $U \in \mathcal{U}_n^{r}$, then we put $\varphi(x) = \{r(U)\}$.

Case 4. Finally, we suppose that $U \notin \mathcal{U}_n^{\ell} \cup \mathcal{U}_n^{r}$. Then we put $\varphi(x) = \{a(U)\}$, where $a(U)$ is defined in the proof of Theorem 2.1 in [7]. Then we can see that $\varphi : X \to 2^A$ is upper semicontinuous.

To define an extender $u : C(A \times Y, Z) \to C(X \times Y, Z)$, let $f \in C(A \times Y, Z)$. First, for each n and each $U \in \mathcal{U}_n$ we shall define a continuous function $f_U : U \times Y \to Z$. We consider the following four cases.

Case 1. Suppose that $U \in \mathcal{U}_n^\ell \cap \mathcal{U}_n^r$. If $U = \{x\}$, we define $f_U(x,y) = f(l(U),y)$ for each $y \in Y$. If U contains at least two points, we define

$$f_U(x,y) = \begin{cases} f(l(U),y), & \text{if } x < s(U), \\ (1-\psi_U)(x)\cdot f(l(U),y) + \psi_U(x)\cdot f(r(U),y), & \text{if } s(U) \le x \le t(U), \\ f(r(U),y), & \text{if } x > t(U), \end{cases}$$

for each $(x,y) \in U \times Y$, where $\psi_U : X \to I$ is a continuous mapping such that $(\leftarrow, l(U)] \subset \psi_U^{-1}(0)$ and $[r(U), \to) \subset \psi_U^{-1}(1)$.

Case 2. If $U \in \mathcal{U}_n^\ell$ and $U \notin \mathcal{U}_n^r$, then we put $f_U(x,y) = f(l(U),y)$ for each $(x,y) \in U \times Y$.

Case 3. If $U \notin \mathcal{U}_n^\ell$ and $U \in \mathcal{U}_n^r$, then we put $f_U(x,y) = f(r(U),y)$ for each $(x,y) \in U \times Y$.

Case 4. If $U \notin \mathcal{U}_n^\ell \cup \mathcal{U}_n^r$, $f_U(x,y) = f(a(U),y)$ for each $(x,y) \in U \times Y$.

We define a function $u(f) : X \times Y \to Z$ as follows:

$$u(f)(x,y) = \begin{cases} f(x,y), & \text{if } x \in A, \\ f_U(x,y), & \text{if } x \in U \text{ for some } U \in \mathcal{U}. \end{cases}$$

In a similar fashion to [7], we can see that $u(f)$ is a continuous extension of f and the range of $u(f)$ is contained in the closed convex hull of the range of f.

By use of the upper semicontinuity of φ, we can show that the extender u above is continuous with respect to the point convergence topology, compact-open topology and uniform convergence topology (cf. [8]).

In a similar fashion as the proof of Theorem 2.1, we obtain the following (in fact, the proof of this case is more simple than Theorem 2.1).

Theorem 2.2. *Let X be a GO-space, A a closed subspace of X and $X - A = \bigcup \mathcal{U}$, where $\mathcal{U}$ is a disjoint family of convex components of $X - A$. If $\mathcal{U}' = \{U \in \mathcal{U} :$ U has neither $l(U)$ nor $r(U)\}$ is discrete in X, then A is continuously π_D-embedded in X.*

Corollary 2.1. *Let X be a locally compact GO-space. Then every closed subspace A of X is continuously π_D-embedded in X.*

Corollary 2.2. *Every closed subspace of the Sorgenfrey line $\mathbb{S}$ is continuously π_D-embedded.*

Corollary 2.3. *Let X be a GO-space such that the underlining ordered set is well-ordered. Then every closed subspace A of X is continuously π_D-embedded.*

Now, we have the following corollaries.

Corollary 2.4. *Let $X_i (i = 1, 2, \cdots n)$ be perfectly normal GO-spaces with $E(X_i)$ σ-discrete in X_i and A_i are closed subsets in X_i. Then, $\prod_{i=1}^n A_i$ is D-embedded in $\prod_{i=1}^n X_i$.*

Corollary 2.5. *Let κ be an ordinal and $A_i (i = 1, 2, \cdots, n)$ are closed subsets of κ. Then $\prod_{i=1}^n A_i$ is D-embedded in κ^n.*

Remark. In [5], Heath and Lutzer proved that for every closed subspace A of a GO-space X there is a simultaneous extender $u : C^*(A) \to C^*(X)$. However, Heath, Lutzer and Zenor [6] proved that there is no Dugundji extender $u : C^*(\mathbb{Q}) \to C^*(\mathbb{M})$ which is continuous when both function spaces are equipped with the compact-open topology nor the pointwise convergence topology, where $\mathbb{M}$ is the Michael line and $\mathbb{Q}$ is the subspace of $\mathbb{M}$ consisting of all rationals.

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