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ON THE SHAPE ASPHERICITY OF COMPACTA

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1. INTRODUCTION

Daverman [Da_2] introduced the notion of (strongly) hereditarily aspherical compacta, which is a natural extension of the usual asphericity. A compactum is *hereditarily aspherical* if each closed subset A is k-UV for all $k > 1$. Saying A is k-UV means that under some embedding of A in the Hilbert cube, each neighborhood U of A contains a neighborhood V of A such that all maps $S^k \to V$ extends to maps $B^{k+1} \to U$. Of course, this k-UV property is independent of the embedding of A in metrizable ANR's. X is said to have Property UV^n if it has Property k-UV for $0 \leq k \leq n$. A compactum X is said to be *strongly hereditarily aspherical* if X can be embedded in the Hilbert cube Q such that for each $\varepsilon > 0$ there exists an ε-cover $\mathcal{U}$ of X by open collection of Q, where the union of any subcollection of elements of $\mathcal{U}$ is aspherical. Note that strongly hereditary asphericity implies hereditary asphericity. Manifolds and complexes of dimension 3 and higher cannot possess such a hereditary property. Surprisingly, Daverman [Da_2] constructed examples of strongly hereditarily aspherical 3- and 4-dimensional compacta and hereditarily aspherical generalized 3-dimensional manifold. Recently, Daverman and Dranishnikov [Da-Dra] proved the existence of strongly hereditarily aspherical compacta of arbitrary dimension. Furthermore, Daverman [Da_2] proved that the class of strongly hereditarily aspherical compacta contains all 2-dimensional compacta of rational cohomological dimension one and cell-like maps defined either on strongly hereditarily aspherical compacta or on locally simply connected hereditarily aspherical compacta cannot raise dimension.

Dydak and the author [D-Y] introduced a different generalization of asphericity: let us recall a compactum X *shape aspherical* if any map $f: X \to P$ from X to a polyhedron P factors up to homotopy as $f \simeq h \circ g$, where $g\colon X \to K$, $h\colon K \to P$ and K is an aspherical CW-complex (i.e., K is a $K(G,1)$). Obviously, shape aspherical compacta are aspherical and strongly hereditarily aspherical compacta are hereditarily shape aspherical. They modified several results known to be true for the class of strongly hereditarily aspherical compacta.

In this paper, we shall investigate the fundamental properties of the shape asphericity which is natural from the view point of homotopy theory (see [Y] for the details).

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2. Hereditarily aspherical compacta

In this section we shall introduce the following results in the joint work [D-Y] with Jerzy Dydak (University of Tennessee, Knoxville).

Theorem 2.1. *If $X \in LC^1$ is a hereditarily aspherical compactum, then $X \in ANR$. In particular, X is strongly hereditarily aspherical.*

Remark 2.2. Theorem 2.1 gives a partial answer to Question 3 in [Da_2]. Since the property of the hereditary asphericity and of being LC^1 are preserved on a cell-like map, Theorem 9 in [Da_2] follows from Theorem 2.1.

Remark 2.3. Daverman [Da_2] constructed a hereditarily aspherical generalized 3- dimensional manifold. Does there exists a hereditarily aspherical ANR of dimension greater than 3 ?

Theorem 2.4. *Suppose $f : X \to Y$ is a cell-like map of compacta and $f^{-1}(A)$ is shape aspherical for each closed subset A of Y. Then*
1. Y is hereditarily shape aspherical,
2. f is a hereditary shape equivalence,
3. $\dim X \geq \dim Y$.

Theorem 2.5. *Suppose G is a group containing integers. If $\dim X \leq 2$ and $\dim_G X = 1$, then X is hereditarily shape aspherical.*

Theorem 2.6. *Suppose G is a group containing integers. Then the following conditions are equivalent:*
1. $\dim X \leq 2$ *and* $\dim_G X = 1$,
2. $\dim_{G*_{\mathbf{Z}}G} X = 1$, *where $*_{\mathbf{Z}}$ means the amalgamated free product with* $\mathbf{Z}$.

Definition. We say that a compactum X is of *perfect cohomological dimension 1* provided $\dim_G X = 1$ for all perfect groups G.

Remark. Compacta of perfect cohomological dimension 1 were introduced in [D-R] as Kainian compacta.

Corollary 2.7 (Dranishnikov-Repovš [D-R]). *If X is a compactum of perfect cohomological dimension 1, then* $\dim X \leq 2$.

It is well known that for compacta X and Y the equality

$(*)$ $\dim(X \times Y) = \dim X + \dim Y$

does not generally hold [Bol_1,Bor_1,P]. Therefore it makes sense to study compacta such that the equality $(*)$ holds [Ko_1, Ko_2]. One of the known cases is of compacta X and Y which have the property Δ in the sense of K. Borsuk [Bor_2, p.178]. Since 2-dimensional ANR compacta have the property Δ, the equality $(*)$ holds (in fact, these spaces are dimensionally full-valued [Ko_1]). Boltyanskii [Bol_1] (see also [Bor_1]) constructed 2-dimensional LC^0 compactum with $\dim(X \times X) = 3$. Thus, to consider whether the equality $(*)$ holds in the class of 2-dimensional LC^1 compacta is natural. We shall solve this problem in affirmative.

Definition. Recall that a space is called *semilocally simply connected* provided X has a basis of open sets U such that the inclusion induced homomorphism $\pi_1(U) \to \pi_1(X)$ is trivial.

Theorem 2.8. *Let X be a locally connected and semilocally simply connected compactum of dimension ≥ 2. Then, $\dim_G X > 1$ for all groups $G \neq 1$.*

Corollary 2.9. *Let X be a two-dimensional, locally connected and semilocally simply connected compactum. Then, for any compactum Y*

$$\dim(X \times Y) = \dim X + \dim Y.$$

3. BASIC PROPERTIES OF SHAPE ASPHERICAL COMPACTA

This section presents some basic properties about the shape asphericity.

Proposition 3.1 (c.f. [Da-Dra]). *Let X be a shape aspherical compactum. Then X has property UV^1 if and only if it is cell-like.*

Remark. One may consider whether an aspherical compactum is shape aspherical. But it is not always true. Let T be the Taylor's example [T]. Then T is aspherical (in fact, has property UV^∞), but not shape aspherical, because of the proposition above and non-cell-likeness of it (c.f. [Da-Dra]). Even *finite* dimensional case, asphericity and shape asphericity do not coincide in general (for example, consider the Daverman-Dranishnikov's example F in [Da-Dra, Theorem 3.7] and use Theorem 3.4 in [D-Y]).

Every morphism $F\colon (X,*) \to (Y,*)$ of the pointed shape category induces a well-defined morphism pro-$\pi_1(F)$: pro-$\pi_1(X,*) \to$ pro-$\pi_1(Y,*)$. In fact, it follows from the following theorem that a shape aspherical continuum is determined up to shape type by its pro-fundamental group. We denote the set of all morphisms (in the category of pro-groups) from pro-$\pi_1(X,*)$ to pro-$\pi_1(Y,*)$ by pro-$\pi_1((X,*),(Y,*))$.

Theorem 3.2. *Let $(X,*)$ be a pointed continuum, $(Y,*)$ a pointed shape aspherical continuum. Then we have the following bijection:*

$$Sh((X,*),(Y,*)) \approx \textit{pro-}\pi_1((X,*),(Y,*)).$$

Corollary 3.3. *Two shape aspherical pointed continua having isomorphic pro-fundamental groups have the same shape type.*

Corollary 3.4. *If X and Y are shape aspherical pointed continua with isomorphic pro-fundamental groups, then pro-$H_q(X,G) \approx$ pro-$H_q(Y,G)$ and pro-$H^q(X,G) \approx$ pro-$H^q(Y,G)$ for any abelian group G.*

From the Smith's theorem [Sm, p.367], we know that a finite dimensional aspherical complex has fundamental group, which contains no elements of finite order. Motivated by the result, we introduce a new notation. An inverse sequence $\{G_i, g^i_{i-1}\}$ of groups is *pro-torsion free* if for each $i \in \mathbf{N}$ there is a $j > i$ such that $\{g \in G_j :$ g is an element of finite order$\} \subseteq \mathrm{Ker}(g^j_i)$, (the definition coincides the usual one in the category of abelian groups). Then we naturally pose the following question:

Question. Does a finite dimensional shape aspherical compactum have pro-fundamental group pro-torsion free?

Remark. Examples constructed by Daverman, Daverman-Dranishnikov [Da_2, Da-Dra] have pro-fundamental groups pro-torsion free.

If a non-trivial group G is not torsion free, we can find a non-trivial finite cyclic subgroup of G. Suppose $\{G_i, p^i_{i-1}\}$ is not pro-torsion free. Then we have a non-trivial pro-subgroup $\{F_i, p^i_{i-1}|_{F_i}\}$ of $\{G_i\}$, where F_i is the subgroup of G_i generated by the set of all finite order elements of G_i. We may not obtain a non-trivial pro-subgroup $\{\mathbf{Z}_{d_i}\}$ of $\{F_i\}$ consisting of finite cyclic groups (for example, consider the pro-group $\{G_i, p^i_{i-1}\}$ defined by $G_i = \prod_{j\in\mathbf{N}} G_{ij}$ and $p^i_{i-1} = \prod_{j\in\mathbf{N}} p(j)^i_{i-1}$, where $G_{ij} = \mathbf{Z}_2$ and $p(j)^i_{i-1} = id_{\mathbf{Z}_2}$ if $j \geq i$ or trivial if $j < i$), but we shall give examples, which suggest that the question will be affirmative in the case.

Example 3.5. Let n be an odd prime and $\{\mathbf{Z}_n \xleftarrow{f^2_1} \mathbf{Z}_{2n} \xleftarrow{f^3_2} \mathbf{Z}_{4n} \leftarrow \cdots\}$ a (non-trivial) pro-group consisting of finite (multiplicatively written) cyclic groups $\mathbf{Z}_{2^{k-1}n} = \{g^0_k = 1_k, g_k, \cdots, g_k^{2^{k-1}n-1}\}$ with generators g_k of the order $2^{k-1}n$ and homomorphisms $f^k_{k-1}\colon \mathbf{Z}_{2^{k-1}n} \to \mathbf{Z}_{2^{k-2}n}$ induced by $f^k_{k-1}(g_k) = g_{k-1}$. Then there exist no finite dimensional shape aspherical compacta with pro-fundamental group which is isomorphic to the pro-group.

Remark. Since the example above essentially contains a pro-group $\{\mathbf{Z} \leftarrow \mathbf{Z} \leftarrow \cdots\}$, we may see the same conclusion using an argument of the following example. Also we note that there is an example in which the proof above breaks down (calculate the odd-dimensional pro-homology groups of $\{\mathbf{Z}_p \leftarrow \mathbf{Z}_{p^2} \leftarrow \cdots\}$, where p is a prime).

More generally, we have the following example along a proof of the Smith's theorem ([K-N-S]).

Example 3.6. Let $\{G_i, p^i_{i-1}\}$ be a pro-group containing the pro-group $\{\mathbf{Z}_{2^{k-1}n}, f^k_{k-1}\}$ in the example 3.5 as a pro-subgroup. Then there exist no finite dimensional shape aspherical continua with pro-fundamental group which is isomorphic to the pro-group.

Theorem 3.7. *Let X_i, $i \in \mathbf{N}$, be compacta. Then the following are equivalent:*

(1) *the product space $\prod_i X_i$ is shape aspherical,*
(2) *each X_i are shape aspherical.*

A map $f\colon X \to Y$ is *refinable* if for every $\varepsilon > 0$ there is an ε-map $g\colon X \to Y$ such that $d(f,g) < \varepsilon$. Note that refinable maps do not preserve shapes in general.

Theorem 3.8. *Let $f\colon X \to Y$ be a refinable map between compacta. Then*

(1) *X is shape aspherical if and only if so is Y,*
(2) *if X is hereditarily shape aspherical and Y is LC^1, f is hereditarily shape equivalence.*

4. Shape asphericity pasting theorem

J. H. C. Whitehead [WHJ$_1$] proved an "addition theorem" of aspherical polyhedra with an application to the study of knots and linkages as an answer of question proposed by Eilenberg. In this section, we shall show a similar one of shape aspherical compacta under some conditions.

An inverse system $\{G_i, g^i_{i-1}\}$ of groups and homomorphisms is *pro-abelian* if for each $i \in \mathbf{N}$ there is a $j > i$ such that $\operatorname{Im} g^j_i \subseteq G_i$ is abelian. Then note that there is a pro-group $\{H_i, h^i_{i-1}\}$ such that each H_i are abelian groups and $\{G_i, g^i_{i-1}\} \approx \{H_i, h^i_{i-1}\}$. A pointed compactum X has pro-$\pi_1(X,*)$ pro-abelian if whenever X is written as an inverse limit $X \cong \varprojlim\{(P_i,*), f^i_{i-1}\}$ of finite pointed CW-complexes, the system $\{\pi_1(P_i,*), f^i_{i-1}\}$ is pro-abelian. Then if X is a compactum with pro-$\pi_1(X,*)$ pro-abelian embedded in the Hilbert cube Q as a Z-set, then X has a basis of open neighborhoods consisting of sets with abelian fundamental groups (see the proof of [D-Y, Lemma 3.2]).

Theorem 4.1. *Suppose X_1, X_2 and $X_0 = X_1 \cap X_2$ are connected, shape aspherical compacta such that pro-$\pi_1(X_0,*) \to$ pro-$\pi_1(X_i,*)$ is a monomorphism $(i = 1,2)$ for some base point $* \in X_0$. If X_i, $i = 1,2$, have pro-$\pi_1(X_i,*)$ pro-abelian, then $X_1 \cup X_2$ is shape aspherical.*

5. Generalized Whitehead conjecture

The following conjecture is proposed by Daverman-Dranishnikov [Da-Dra], as a generalization of the classical Whitehead conjecture, for an approach of their problems.

Generalized Whitehead conjecture. *If X is a 2-dimensional compactum having Property 2-UV, then every subcompactum has Property 2-UV.*

We shall touch upon the conjecture above, under some condition, along a technique of the classical Cockcroft's paper [Cc].

Let $\{(K_i,*), f^{i+1}_i\}$ be an inverse system of finite polyhedra with the limit $(X,*)$. For a pointed map $p_i\colon K_i \to K(\pi_1(K_i,*),1)$ from K_i to an Eilenberg-MacLane complex of type $(\pi_1(K_i,*),1)$ with an isomorphism $\pi_1(p_i)$, we have a map $\tilde{f}^{i+1}_i\colon K(\pi_1(K_{i+1},*),1) \to K(\pi_1(K_i,*),1)$ such that $p_i \circ f^{i+1}_i \simeq \tilde{f}^{i+1}_i \circ p_{i+1}$:

$$(*) \qquad \begin{array}{ccc} K_i & \xleftarrow{f^{i+1}_i} & K_{i+1} \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_{i+1}} \\ K(\pi_1(K_i,*),1) & \xleftarrow{\tilde{f}^{i+1}_i} & K(\pi_1(K_{i+1},*),1) \end{array}$$

Then we have a pro-group $\{H_n(K(\pi_1(K_i,*),1)), H_n(\tilde{f}^{i+1}_i)\}$ over the homology groups. Since the pro-group is independent to a resolution of X, we may denote the equivalence class of pro-groups which contains the pro-group by $H_n(\text{pro-}\pi_1(X,*))$.

The following Lemma, as a generalization of the Hopf theorem [Ho$_1$, Ho$_2$], is useful for us.

Lemma 5.1. *Let X be a pointed continuum.*

(1) *There is an exact sequence:*

$$pro\text{-}\pi_2(X) \xrightarrow{\mathfrak{h}} pro\text{-}H_2(X) \to H_2(pro\text{-}\pi_1(X,*)) \to *,$$

where $\mathfrak{h}$ is the Hurewicz's homomorphism.

(2) *If X is 2-shape dimensional, there are a pro-group $pro\text{-}\Gamma_2(X)$, a monomorphism $pro\text{-}\Gamma_2(X) \to pro\text{-}\pi_2(X)$ and an epimorphism $pro\text{-}\Gamma_2(X) \to H_3(pro\text{-}\pi_1(X,*))$.*

We call that the rank of a pro-group $\mathcal{G} = \{G_i, g_i^{i+1}\}$ with pro-finitely generated abelian is at most n (denoted by $\operatorname{rank}\mathcal{G} \le n$) provided that for every $i \in \mathbf{N}$ there is a $j > i$ such that $\operatorname{Im} g_i^j$ is a finitely generated abelian group with $\operatorname{rank}\operatorname{Im} g_i^j \le n$. We easily see that an epimorphism between the pro-groups preserves the rank and a monomorphism raises the one.

Proposition 5.2. *Let X be a 2-(shape) dimensional aspherical pointed continuum with $pro\text{-}\pi_1(X)$ pro-free abelian. Then we have* $\operatorname{rank}(pro\text{-}\pi_1(X)) \le 2$.

We denote the free abelian group $\oplus_{i=1}^n \mathbf{Z}\langle g_i\rangle$ of rank n by $\mathbf{Z}^n$. Then we know that the i-th homology group of $\mathbf{Z}^n$ is the free abelian group $\bigoplus_{1\le j_1<\cdots<j_i\le n} \mathbf{Z}\langle g_{j_1}, \cdots, g_{j_i}\rangle$ of rank $n!/i!(n-i)!$.

Lemma 5.3. *Let $f\colon \mathbf{Z}^n \to \mathbf{Z}^m$ be a homomorphism between free abelian groups of rank n and m represented by the following matrix*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Then the homomorphisms between the 2- and 3-dimensional homology groups induced by f are represented by

$$f_{*2}(\langle g_a, g_b\rangle) = \sum_{1\le p<q\le m} \begin{vmatrix} a_{pa} & a_{pb} \\ a_{qa} & a_{qb} \end{vmatrix} \langle h_p, h_q\rangle \textit{ for } 1 \le a < b \le n,$$

$$f_{*3}(\langle g_a, g_b, g_c\rangle) = \sum_{1\le p<q<r\le m} \begin{vmatrix} a_{pa} & a_{pb} & a_{pc} \\ a_{qa} & a_{qb} & a_{qc} \\ a_{ra} & a_{rb} & a_{rc} \end{vmatrix} \langle h_p, h_q, h_r\rangle \textit{ for } 1 \le a < b < c \le n.$$

By using lemma 5.1, proposition 5.2 and lemma 5.3, we can show the following theorem.

Theorem 5.4. *Let X be a pointed 2-dimensional continuum having Property 2-UV with pro-$\pi_1(X,*)$ pro-free abelian and Y a pointed 2-dimensional subcontinuum of X with pro-$\pi_1(Y,*)$ pro-free abelian. Then we have the following table:*

rank *pro*-$\pi_1(X)$ / rank *pro*-$\pi_1(Y)$	2	1	0
≥ 3	/	/	/
2	○ +1-*shape connected*	/	/
1	?	○ +1-*shape connected*	?
0	○	○	○

where the symbol / means that the case does not rise, the symbol ○ does that Y is 2-UV and "+1-shape connected" does that "under the condition (X,Y) is 1-shape connected".

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