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Towards a Non-linear Extension of Stochastic Calculus

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Introduction

In a series of recent works [3], [4], [5], [6] a new approach to quantum stochastic calculus has been launched out through the study of stochastic limit of quantum theory. This new approach is not based upon a quantum Itô theory but upon a new type of distribution theory on simplexes and reveals intrinsic features of quantum white noise. Meanwhile, independently of that, quantum stochastic processes have been discussed on the basis of white noise (Hida) distribution theory [11], [12], [19], [20], [21]. This approach seems useful for studying differential equations for Fock space operators involving higher powers of quantum white noises, at least for the existence and uniqueness of a solution in a distribution sense [22]. Thus it is quite natural to ask what can be said by combining the above two ideas together. In the present paper we discuss the simplest problem of a unitary equation driven by $b_t^{\dagger 2}$ and b_t^2, which is already beyond the reach of the Itô or Hudson–Parthasarathy stochastic calculus. Although our argument stays at somehow formal level, it is most plausible that we are led along a new direction towards a non-linear stochastic calculus.

1 Classsical and quantum stochastic calculi

Classsical stochastic calculus was initiated by Itô [14]. It gives a meaning to equations of the form

$$df(X_t) = Zf(X_t)dt + Lf(X_t)dW_t, \tag{1.1}$$

where $W = W_t$ is the standard $\mathbb{R}^d$-valued Brownian motion starting at zero, L and Z are respectively a second order and a first order differential operators on $\mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Here the Schwartz test function space can be replaced with a larger function space but for the moment we stick to this choice. Equation (1.1) is interpreted as a symbolic notation for the integral equation:

$$f(X_t) = f(X_0) + \int_0^t Zf(X_s)ds + \int_0^t Lf(X_s)dW_s, \tag{1.2}$$

where the second integral is a stochastic integral. Equation (1.2) can be shown to admit a unique solution for any initial condition of the form $X(0) = X_0$, where X_0 is a random variable wich is measurable with respect to a σ-algebra independent of that of W.

Quantum stochastic calculus was initiated by Hudson and Parthasarathy [13]. It gives a meaning to equations of the form

$$dU_t = \left(DdB_t^\dagger - D^\dagger dB_t + \left(-\frac{\gamma}{2}D^\dagger D + iH\right)dt\right)U_t, \qquad U_0 = I, \tag{1.3}$$

where the pair $(B_t^\dagger, B_t)$ is the Fock Brownian motion with variance $\gamma > 0$ acting on the Boson Fock space $\Gamma(L^2(\mathbb{R}))$, and D, $H = H^*$ are operators on a Hilbert space $\mathcal{H}_S$. Thus U_t will be an operator acing on $\mathcal{H}_S \otimes \Gamma(L^2(\mathbb{R}))$. More generally, one can replace $L^2(\mathbb{R})$ by $L^2(\mathbb{R}) \otimes \mathcal{K}$, where $\mathcal{K}$ is any Hilbert space, see e.g., [17], [23]. Equation (1.3) is interpreted as a symbolic notation for the integral equation

$$U_t = 1 + \int_0^t \left(DdB_s^\dagger - D^\dagger dB_s\right)U_s + \int_0^t \left(-\frac{\gamma}{2}D^\dagger D + iH\right)ds\, U_s, \tag{1.4}$$

where the first integral is a quantum stochastic integral. Equation (1.3) can be shown to admit a unique solution which is a unitary operator on the space $\mathcal{H}_S \otimes \Gamma(L^2(\mathbb{R}))$ whenever the operators D and H satisfy some conditions.

Quantum stochastic calculus includes the classical one. Let $(\Omega, P, \mathcal{F})$ be the probability space of W. Then each random variable of the process X_t defines an homomorphism j_t : $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \otimes L^\infty(\Omega, P, \mathcal{F})$ by

$$j_t(f) = f(X_t). \tag{1.5}$$

The family of homomorphisms j_t characterizes the process and constitutes a natural algebraic characterization of it. On the other hand, the classical Brownian motion W can be realized as an operator process (i.e. $W_t = B_t + B_t^\dagger$) on the Fock space $\Gamma(L^2(\mathbb{R}))$, $\mathcal{H}_S$ is a Hilbert space corresponding to a space of possible initial conditions for the Itô equation (1.1), the process $f(X_t)$ can be thought as a multiplication operator on the space $\mathcal{H}_S \otimes \Gamma(L^2(\mathbb{R}))$ and the classical flow j_t, defined by (1.5), is given by

$$j_t(f) = f(X_t) = U_t^* f(X_0) U_t,$$

where U_t is the solution of a QSDE of the form (1.3) for an appropriate choice of the coefficients operator D, see [9], [10], [23]. Thus, the flow j_t plays a crucial role in the link between classical and quantum stochastic processes, see e.g., [7].

2 Stochastic limit

As is clear from the previous section, for a generalization of stochastic calculus fundamental object to start with is the flow j_t or the associated one-parameter family of unitary operators U_t. The stochastic limit of quantum theory developed recently in [6] suggests a new possibility of constructing unitary operators U_t.

The starting point of this theory is not a stochastic equation but a usual Schrödinger equation in interaction representation:

$$\partial_t U_t = -i\lambda H_I(t) U_t, \qquad U_0 = I, \tag{2.1}$$

where $\lambda > 0$ is a coupling constant. For example, $H_I(t)$ is given as

$$H_I(t) = e^{itH_0}(D \otimes A^\dagger(g) + D^\dagger \otimes A(g))e^{-itH_0}, \qquad H_0 = H_{\text{system}} + H_{\text{reservoir}},$$

which is known as a *Hamiltonian of laser type.* A simple case would be

$$H_I(t) = D \otimes A_t^\dagger + D^\dagger \otimes A_t, \qquad A_t = A(S_t g).$$

The (formal) solution of (2.1) is given by the iterated series:

$$\begin{aligned} U_t &= I + (-i\lambda)\int_0^t dt_1 H_I(t_1)U_{t_1} \\ &= I + \sum_{n=1}^{\infty}(-i\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)\cdots H_I(t_n). \end{aligned}$$

Particularly interesting is the scaling limit of (2.1) according to the law $t \to t/\lambda^2$, which is motivated both by mathematics (central limit theorem) and by physics (Friedrichs–van Hove rescaling). The rescaled solution is

$$\begin{aligned} U_{t/\lambda^2}^{(\lambda)} &= I + \sum_{n=1}^{\infty}(-i\lambda)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)\cdots H_I(t_n) \\ &= I - i\lambda \int_0^{t/\lambda^2} dt_1 H_I(t_1) - \lambda^2 \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 H_I(t_1)H_I(t_2) + \cdots \end{aligned} \tag{2.2}$$

and the rescaled equation is of the form:

$$\partial_t U_{t/\lambda^2}^{(\lambda)} = -i(D \otimes A_t^{(\lambda)\dagger} + D^\dagger \otimes A_t^{(\lambda)})U_{t/\lambda^2}^{(\lambda)}, \qquad U_0^{(\lambda)} = I. \tag{2.3}$$

It was proved in [3] that the iterated series solution (2.2) converges as $\lambda \to 0$ to the solution of the quantum stochastic differential equation (1.3) with $H = \kappa D^\dagger D$, $\kappa \in \mathbb{R}$, in the sense of quantum convergence in law.

More recently, it has been proved in [5] that the iterated series solution (2.2) converges term by term to the same limit in the same sense as above to the iterated series solution of the distribution equation

$$\partial_t U_t = -i(D \otimes b_t^\dagger + D^\dagger \otimes b_t)U_t, \qquad U_0 = I, \tag{2.4}$$

where $b_t^\dagger$, b_t are the annihilation and creation operators of the Boson Fock white noise with variance

$$\gamma = \int_{-\infty}^{\infty} \langle g, S_t g\rangle\, dt > 0.$$

It is known that $b_t^\dagger$, b_t are characterized (up to unitary equivalence) by the algebraic relations

$$\begin{aligned} [b_t, b_s^\dagger] &= \gamma\delta(s-t), \quad t, s \in \mathbb{R}, \\ b_t \Phi &= 0, \quad t \in \mathbb{R}, \end{aligned} \tag{2.5}$$

where Φ is the Fock vacuum.

3 The white noise approach to stochastic calculus

Equation (2.4) has a well-defined meaning in terms of matrix elements in the number vectors. Let us recall from [5] how to give an intrinsic meaning to equation (2.4). This shall be useful for comparison with the nonlinear extension of this equation to be discussed below.

The number vectors are defined by the realtion

$$b_{t_1}^\dagger \cdots b_{t_k}^\dagger \Phi = |t_1, \cdots, t_k\rangle, \tag{3.1}$$

which has to be interpreted in the distribution sense, i.e. for any test functions $\phi_1, \cdots, \phi_k \in \mathcal{S}(\mathbb{R})$, one defines

$$b^\dagger(\phi_1) \cdots b^\dagger(\phi_k)\Phi = \int dt_1 \cdots \int dt_k \, \phi_1(t_1) \cdots \phi_k(t_k)|t_1, \cdots, t_k\rangle. \tag{3.2}$$

In what follows we put $D = I$ for simplicity. Then (2.4) becomes

$$\partial_t U_t = -i(b_t^\dagger + b_t)U_t, \qquad U_0 = I, \tag{3.3}$$

and its weak formulation is

$$\begin{aligned} \partial_t\langle s_1, \cdots, s_h|U_t|t_1, \cdots, t_k\rangle &= -i\langle s_1, \cdots, s_h|(b_t + b_t^\dagger)U_t|t_1, \cdots, t_k\rangle \\ &= -i\sum_{j=1}^h \gamma\delta(t - s_j)\langle s_1, \cdots, \widehat{s}_j, \cdots, s_h|U_t|t_1, \cdots, t_k\rangle \\ &\qquad -i\langle t, s_1, \cdots, s_h|U_t|t_1, \cdots, t_k\rangle \end{aligned} \tag{3.4}$$

The right hand side of equation (3.4) is still a distribution. To reduce everything to ordinary functions one interprets (3.3) as the integral equation

$$U_t = I - i\int_0^t (b_s^\dagger + b_s)U_s ds. \tag{3.5}$$

Then

$$\begin{aligned} &\langle s_1, \cdots, s_h|U_t|t_1, \cdots, t_k\rangle = \\ &\quad = \langle s_1, \cdots, s_h|t_1, \cdots, t_k\rangle \\ &\qquad -i\sum_{j=1}^h \gamma\int_0^t \delta(s - s_j)\langle s_1, \cdots, \widehat{s}_j, \cdots, s_h|U_s|t_1, \cdots, t_k\rangle ds \\ &\qquad -i\int_0^t \langle s, s_1, \cdots, s_h|U_s|t_1, \cdots, t_k\rangle ds \\ &\quad = \langle s_1, \cdots, s_h|t_1, \cdots, t_k\rangle \\ &\qquad -i\gamma\sum_{j=1}^h \chi_{[0,t]}(s_j)\langle s_1, \cdots, \widehat{s}_j, \cdots, s_h|U_{s_j}|t_1, \cdots, t_k\rangle \\ &\qquad -i\int_{\mathbb{R}} \chi_{[0,t]}(s)\langle s, s_1, \cdots, s_h|U_s|t_1, \cdots, t_k\rangle ds. \end{aligned} \tag{3.6}$$

In view of $U_t = \langle \partial_s U_s, \chi_{[0,t]}(s)\rangle$, we replace $\chi_{[0,t]}$ in (3.6) with $\phi \in \mathcal{S}(\mathbb{R})$ to obtain

$$\begin{aligned}
&\langle s_1, \cdots, s_h| \langle \partial_s U_s, \phi\rangle |t_1, \cdots, t_k\rangle = \\
&\quad = \langle s_1, \cdots, s_h|t_1, \cdots, t_k\rangle \\
&\qquad -i\gamma \sum_{j=1}^{h} \phi(s_j)\langle s_1, \cdots, \widehat{s_j}, \cdots, s_h|U_{s_j}|t_1, \cdots, t_k\rangle \\
&\qquad -i\int_{\mathbb{R}} \phi(s)\langle s, s_1, \cdots, s_h|U_s|t_1, \cdots, t_k\rangle ds.
\end{aligned}$$

So the equation (3.3) has an meaning weakly on number vectors. A similar discussion can be done for the exponential vectors.

In order to identify the distribution equation (3.3) with a quantum stochastic equation, it is convenient to write equation (3.3) in normal form. Since

$$\partial_t U_t = -i(b_t^\dagger U_t + U_t b_t + [b_t, U_t]),$$

in order to bring equation (2.4) into a normal form we have to calulate the commutator $[b_t, U_t]$. This is done through the following result [1], [6].

Theorem 3.1 *If U_t is a solution of (1.3) with $H = \kappa D^\dagger D$, then*

$$[b_t, U_t] = \gamma_- D U_t, \qquad \gamma_- = \frac{\gamma}{2} - i\kappa.$$

where the identity is understood weakly on the exponential or number vectors.

The proof of Theorem 3.1 is based upon two fundamental principles of the quantum white noise theory on the standard simplex established in [5].

- **The causal commutator rule** When the operators b_t, $b_t^\dagger$ in the iterated series for U_t are brought into a normal ordered form, we must not use the usual commutation relation (2.5) but the *causal commutation relation* defined by

$$[b_\sigma, b_\tau^\dagger] = \gamma_- \delta_+(\sigma - \tau), \qquad \sigma \geq \tau, \tag{3.7}$$

  where γ_- is a complex number and δ_+ is the causal delta function on a simplex. Notice that the causal commutator is defined only for σ (the time index of annihilator) greater than τ (the time index of creator). So the causal commutator rule should not be interpreted as a new commutation relation, but only as a rule to bring in normal order the creation and annihilation operator which appear in the iterated series.

- **The principle of consecutive times** Any commutator of the form

$$\int_0^t [b_t, U_s]\, ds$$

  vanishes identically. This is, in fact, a corollary of the principle of consecutive times but is all we shall need in the present paper.

From Theorem 3.1 it follows that the normal form of (3.3) is given by

$$\partial_t U_t = -i(b_t^\dagger U_t + U_t b_t + \gamma_- U_t). \tag{3.8}$$

Similarly, the normal form of (2.4) is given as

$$\partial_t U_t = -i(D b_t^\dagger U_t + D^\dagger U_t b_t + \gamma_- D^\dagger D U_t), \tag{3.9}$$

where short hand notation such as $D = D \otimes I$, $b_t = I \otimes b_t$ is used. This result solves the above conjecture affirmatively. In fact, by taking matrix elements in the exponential vectors of both sides of the two equations (3.3) and (3.8) one easily obtains the same ordinary differential equation.

The above argument suggests a new approach to stochastic calculus based directly on white noise and not through the intermediate Brownian motion. The basic idea of this approach can be formulated as follows:

1. Write an equation in normal form:

$$\partial_t U_t = A b_t^\dagger U_t + B U_t b_t + C U_t, \qquad U_0 = I, \tag{3.10}$$

   where A, B, C are system operators.

2. Prove the existence and uniqueness of the solution in terms of operator symbols

$$q_t(f, g) = \langle \psi_f, U_t \psi_g \rangle ,$$

   where ψ_f, ψ_g are exponential vectors.

3. Prove a regularity condition showing that the solution of equation (3.10) is a *bona fide* operator and not only a distribution.

4. Formulate the unitarity condition for U_t, namely, $U_t^* U_t = U_t U_t^* = I$, and prove it.

In the paper [5] this program was realized for the usual (classical and quantum) stochastic calculi. The advantage of this formulation with respect to the traditional ones lies in the fact that it naturally suggests the problem of extending equation (3.3) to the case of an equation which is still normally ordered, but now depending on higher powers of the white noise $b_t^\dagger, b_t$. Such equations had already been studied in the theory of white noise and existence and uniqueness theorems in the space of Hida distributions are already available [22]. It is therefore a natural program to combine the result of [22] with the above mentioned ones and to investigate regularity and unitarity.

4 Non-linear white noise stochastic calculus

In order to realize the program described at the end of the previous section we have to overcome two competing requirements. On the one hand, we would like to start with an equation in a normally ordered form. On the other hand, we come to a non-trivial task of recognizing the unitarity condition in an equation in normal form. The advantage of the singular Hamiltonian formulation of equation (2.4) is that the formal unitarity condition is automatically satisfied. Namely, one can hope that, if one starts from a formally unitary equation and then puts it in normal order, one obtains an equation whose solution is effectively unitary. We are more interested in realizing this program in the non-linear case. Again we can apply the two basic principles formulated in the previous section, because they are universally valid in the Fock space and do not depend in any way on the linearity of the equation. In fact, they do not depend at all on any particular equation but are intrinsic properties of the Boson Fock white noise calculus.

Thus our starting point shall be the equation

$$\partial_t U_t = -i(b_t^{\dagger 2} + b_t^2)U_t, \qquad U_0 = 1, \tag{4.1}$$

or in the integral form:

$$U_t = 1 - i\int_0^t (b_s^{\dagger 2} + b_s^2)U_s ds. \tag{4.2}$$

Our first goal is to give a meaning to it. Contrary to the linear case, equation (4.2) has no meaning even weakly on the number vectors. In fact, if we calculate formally the matrix elements of the both sides of (4.2) with respect to two number vectors, we obtain the expression:

$$\begin{aligned}
&\langle s_1, \cdots, s_h | U_t | t_1, \cdots, t_k \rangle = \\
&\quad = \langle s_1, \cdots, s_h | t_1, \cdots, t_k \rangle \\
&\quad -i \sum_{\alpha \neq \beta} \gamma^2 \int_0^t \delta(s - s_\alpha)\delta(s - s_\beta)\langle s_1, \cdots, \widehat{s_\alpha}, \cdots, \widehat{s_\beta}, \cdots, s_h | U_s | t_1, \cdots, t_k \rangle ds \\
&\quad -i \int_0^t \langle s, s, s_1, \cdots, s_h | U_s | t_1, \cdots, t_k \rangle ds,
\end{aligned}$$

which involves the ill-defined product of two δ-functions.

In order to give a meaning to equation (4.1) we adopt a different approach, which can be considered as an elaborated version of the scheme of thought one applies when, in usual distribution theory, one wants to give a meaning to the derivative of a non-differentiable function. We use the commutation relations to write equation (4.1) in normal form. In doing so we shall meet some ill-defined quantities such as $\delta(0)$ — the δ-function evaluated at zero! However, as we shall see below these ill-defined quantities appear only as formal additive constants in the Hamiltonian equation and therefore can be eliminated by a subtraction procedure which is well known in physics under the name of *renormalization.* Once written in this form and once the ill-defined quantities have been subtracted, the equation has a weak meaning on the number vectors. We take this normal form as the *definition* of the a priori meaningless equation (4.1).

Lemma 4.1 *Any (formal) solution of equation (4.1) satisfies the following commutation relations:*

$$[b_t, U_t] = -2i\gamma_- b_t^\dagger U_t \tag{4.3}$$

$$[b_t, U_t^*] = 2i\gamma_- U_t^* b_t^\dagger \tag{4.4}$$

$$[U_t^*, b_t^\dagger] = 2i\overline{\gamma_-} U_t^* b_t \tag{4.5}$$

$$[U_t, b_t^\dagger] = -2i\overline{\gamma_-} b_t U_t \tag{4.6}$$

PROOF. Writing equation (4.1) in the integral form:

$$U_t = I - i\int_0^t (b_s^{\dagger 2} + b_s^2) U_s ds,$$

we obtain

$$\begin{aligned}[b_t, U_t] &= [b_t, I] - i\int_0^t [b_t, (b_s^{\dagger 2} + b_s^2)U_s]\, ds \\ &= -i\int_0^t [b_t, (b_s^{\dagger 2} + b_s^2)]U_s\, ds - i\int_0^t (b_s^{\dagger 2} + b_s^2)[b_t, U_s]\, ds. \end{aligned} \tag{4.7}$$

Using

$$[b_t, (b_s^{\dagger 2} + b_s^2)] = [b_t, b_s^{\dagger 2}] = [b_t, b_s^\dagger]b_s^\dagger + b_s^\dagger [b_t, b_s^\dagger] = 2\gamma_- \delta_+(t-s) b_s^\dagger$$

and the time consecutive principle

$$\int_0^t [b_t, U_s] ds = 0,$$

we see that the right hand side of (4.7) becomes

$$= -i\int_0^t 2\gamma_- \delta_+(t-s) b_s^\dagger U_s\, ds = -2i\gamma_- b_t^\dagger U_t.$$

This proves (4.3). The remaining relations are proved similarly. qed

Lemma 4.2 *We have formal identities:*

$$[b_t^2, U_t] = -i4\gamma_- b_t^\dagger U_t b_t - 2i\gamma_-^2 \delta_+(0) U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t \tag{4.8}$$

$$[U_t^*, b_t^{\dagger 2}] = i4\overline{\gamma}_- b_t^\dagger U_t^* b_t + 2i\overline{\gamma}_-^2 \delta_+(0) U_t^* - 4\overline{\gamma}_-^2 U_t^* b_t^2 \tag{4.9}$$

$$[b_t^2, U_t^*] = 4(i\gamma_- - \gamma_-^2) b_t^\dagger U_t^* b_t - 8(|\gamma_-|^2 + i\gamma_-) U_t^* b_t^2 + 2i\gamma_-^2 U_t^* \delta_+(0), \tag{4.10}$$

where $\delta_+(0) = [b_t, b_t^\dagger]$ *is a formal quantity.*

PROOF. Note first that

$$[b_t^2, U_t] = b_t[b_t, U_t] + [b_t, U_t]b_t.$$

Since $[b_t, U_t] = -2i\gamma_- b_t^\dagger U_t$ by Lemma 4.1 (4.3), we obtain

$$\begin{aligned}[b_t^2, U_t] &= b_t(-2i\gamma_- b_t^\dagger U_t) + (-2i\gamma_- b_t^\dagger U_t) b_t \\ &= -2i\gamma_- b_t b_t^\dagger U_t - 2i\gamma_- b_t^\dagger U_t b_t \\ &= -2i\gamma_-([b_t, b_t^\dagger] + b_t^\dagger b_t) U_t - 2i\gamma_- b_t^\dagger U_t b_t. \end{aligned}$$

Introducing a formal quantity

$$\delta_+(0) = [b_t, b_t^\dagger],$$

we continue the calculation:

$$\begin{aligned}
&= -2i\gamma_-(\gamma_-\delta(0) + b_t^\dagger b_t)U_t - 2i\gamma_- b_t^\dagger U_t b_t \\
&= -2i\gamma_-^2\delta(0)U_t - 2i\gamma_- b_t^\dagger b_t U_t - 2i\gamma_- b_t^\dagger U_t b_t \\
&= -2i\gamma_-^2\delta(0)U_t - 2i\gamma_- b_t^\dagger([b_t, U_t] + U_t b_t) - 2i\gamma_- b_t^\dagger U_t b_t \\
&= -2i\gamma_-^2\delta(0)U_t - 2i\gamma_- b_t^\dagger(-2i\gamma_- b_t^\dagger U_t + U_t b_t) - 2i\gamma_- b_t^\dagger U_t b_t \\
&= -2i\gamma_-^2\delta(0)U_t - 4\gamma_-^2 b_t^{\dagger 2}U_t - 2i\gamma_- b_t^\dagger U_t b_t - 2i\gamma_- b_t^\dagger U_t b_t \\
&= -2i\gamma_-^2\delta(0)U_t - 4\gamma_-^2 b_t^{\dagger 2}U_t - 4i\gamma_- b_t^\dagger U_t b_t.
\end{aligned}$$

This proves (4.8) whose adjoint is (4.9).

We then prove (4.10). Note that

$$\begin{aligned}
[b_t, U_t^*] &= 2i\gamma_- U_t^* b_t^\dagger \\
&= 2i\gamma_-([U_t^*, b_t^\dagger] + b_t^\dagger U_t^*) \\
&= 2i\gamma_-(2i\overline{\gamma}_- U_t^* b_t + b_t^\dagger U_t^*) \\
&= -4|\gamma_-|^2 U_t^* b_t + 2i\gamma_- b_t^\dagger U_t^*.
\end{aligned} \tag{4.11}$$

Then

$$\begin{aligned}
[b_t^2, U_t^*] &= b_t[b_t, U_t^*] + [b_t, U_t^*]b_t \\
&= b_t(-4|\gamma_-|^2 U_t^* b_t + 2i\gamma_- b_t^\dagger U_t^*) + (-4|\gamma_-|^2 U_t^* b_t + 2i\gamma_- b_t^\dagger U_t^*)b_t \\
&= -4|\gamma_-|^2 b_t U_t^* b_t + 2i\gamma_- b_t b_t^\dagger U_t^* - 4|\gamma_-|^2 U_t^* b_t^2 + 2i\gamma_- b_t^\dagger U_t^* b_t.
\end{aligned} \tag{4.12}$$

The first two terms are to be computed. In view of (4.11) we have

$$\begin{aligned}
-4|\gamma_-|^2 b_t U_t^* b_t &= -4|\gamma_-|^2([b_t, U_t^*] + U_t^* b_t)b_t \\
&= -4|\gamma_-|^2(-4|\gamma_-|^2 U_t^* b_t + 2i\gamma_- b_t^\dagger U_t^* + U_t^* b_t)b_t \\
&= -4|\gamma_-|^2(1 - 4|\gamma_-|^2)U_t^* b_t^2 - 8i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t
\end{aligned}$$

and

$$\begin{aligned}
2i\gamma_- b_t b_t^\dagger U_t^* &= 2i\gamma_-([b_t, b_t^\dagger] + b_t^\dagger b_t)U_t^* \\
&= 2i\gamma_-(\gamma_-\delta_+(0) + b_t^\dagger b_t)U_t^* \\
&= 2i\gamma_-^2\delta_+(0)U_t^* + 2i\gamma_- b_t^\dagger b_t U_t^* \\
&= 2i\gamma_-^2\delta_+(0)U_t^* + 2i\gamma_- b_t^\dagger([b_t, U_t^*] + U_t^* b_t).
\end{aligned}$$

Using (4.11) to obtain

$$\begin{aligned}
2i\gamma_- b_t b_t^\dagger U_t^* &= 2i\gamma_-^2\delta_+(0)U_t^* + 2i\gamma_- b_t^\dagger(-4|\gamma_-|^2 U_t^* b_t + 2i\gamma_- b_t^\dagger U_t^* + U_t^* b_t) \\
&= 2i\gamma_-^2\delta_+(0)U_t^* + 2i\gamma_-(1 - 4|\gamma_-|^2)b_t^\dagger U_t^* b_t - 4\gamma_-^2 b_t^{\dagger 2}U_t^*.
\end{aligned}$$

Then (4.12) becomes

$$\begin{aligned}
[b_t^2, U_t^*] &= -4|\gamma_-|^2(1 - 4|\gamma_-|^2)U_t^* b_t^2 - 8i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t \\
&\quad +2i\gamma_-^2\delta(0)U_t^* + 2i\gamma_-(1 - 4|\gamma_-|^2)b_t^\dagger U_t^* b_t - 4\gamma_-^2 b_t^{\dagger 2}U_t^* \\
&\quad -4|\gamma_-|^2 U_t^* b_t^2 + 2i\gamma_- b_t^\dagger U_t^* b_t \\
&= -8|\gamma_-|^2(1 - 2|\gamma_-|^2)U_t^* b_t^2 + 4i\gamma_-(1 - 4|\gamma_-|^2)b_t^\dagger U_t^* b_t + 2i\gamma_-^2\delta(0)U_t^*,
\end{aligned}$$

as desired. qed

Proposition 4.3 *The normal form of equation (4.1) is*

$$\partial_t U_t = i(4\gamma_-^2 - 1)b_t^{\dagger 2}U_t - 4\gamma_- b_t^\dagger U_t b_t - iU_t b_t^2 - 2\gamma_-^2\delta_+(0)U_t,$$

whose adjoint form is

$$\partial_t U_t^* = -i(4\overline{\gamma}_-^2 - 1)U_t^* b_t^2 - 4\overline{\gamma}_- b_t^\dagger U_t^* b_t + i b_t^{\dagger 2}U_t^* - 2\overline{\gamma}_-^2\delta_+(0)U_t^*.$$

PROOF. This follows from Lemma 4.2 and the identity

$$\partial_t U_t = -i(b_t^{\dagger 2} + b_t^2)U_t = -ib_t^{\dagger 2}U_t - iU_t b_t^2 - i[b_t^2, U_t].$$

qed

We have thus put equation (4.1) in normal form. Then, as an application of a general theorem on normally ordered equations driven by white noise [22], we can guarantee the existence and uniqueness of its solution, at least in the space of white noise (Hida) distributions. To be more precise, let $(E) \subset \Gamma(L^2(\mathbb{R})) \subset (E)^*$ be the Gelfand triple of white noise functions, for more details see [15], [18].

Lemma 4.4 *Let A, B, C be bounded operators on $\mathcal{H}_S$. Then the differential equation*

$$\partial_t U_t = A b_t^{\dagger 2}U_t + B b_t^\dagger U_t b_t + C U_t b_t^2, \qquad U_0 = I,$$

admits a unique solution in the space $\mathcal{L}(\mathcal{H}_S \otimes (E), \mathcal{H}_S \otimes (E)^*)$.

The proof follows by modifying the result [22, §5] where the initial space $\mathcal{H}_S$ is not taken into account.

5 The normal form of the flow equation

In order to study the isometricity condition for the solution of the equation (4.1), we start with the flow equation

$$j_t(x) = U_t^* x U_t = x_t, \tag{5.1}$$

where x commutes with b_t and $b_t^\dagger$. We need some preliminary lemmas.

Lemma 5.1 *It holds that*

$$U_t^* b_t^\dagger U_t b_t = \frac{1}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{2i\overline{\gamma}_-}{1-4|\gamma_-|^2} U_t^* U_t b_t^2, \tag{5.2}$$

$$b_t^\dagger U_t^* b_t U_t = \frac{1}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t - \frac{2i\gamma_-}{1-4|\gamma_-|^2} b_t^{\dagger 2} U_t^* U_t. \tag{5.3}$$

PROOF. By a direct calculation one obtains

$$\begin{aligned} U_t^* b_t^\dagger U_t b_t &= b_t^\dagger U_t^* U_t b_t + [U_t^*, b_t^\dagger] U_t b_t \\ &= b_t^\dagger U_t^* U_t b_t + 2i\overline{\gamma}_- U_t^* b_t U_t b_t \\ &= b_t^\dagger U_t^* U_t b_t + 2i\overline{\gamma}_- U_t^* ([b_t, U_t] + U_t b_t) b_t \\ &= b_t^\dagger U_t^* U_t b_t + 2i\overline{\gamma}_- U_t^* (-2i\gamma_- b_t^\dagger U_t + U_t b_t) b_t \\ &= b_t^\dagger U_t^* U_t b_t + 4|\gamma_-|^2 U_t^* b_t^\dagger U_t b_t + 2i\overline{\gamma}_- U_t^* U_t b_t^2. \end{aligned}$$

Therefore

$$(1 - 4|\gamma_-|^2) U_t^* b_t^\dagger U_t b_t = b_t^\dagger U_t^* U_t b_t + 2i\overline{\gamma}_- U_t^* U_t b_t^2,$$

which proves (5.2). qed

Lemma 5.2 *Denoting*

$$\alpha = \frac{1}{1 - 4|\gamma_-|^2}$$

one has

$$\begin{aligned} U_t^* b_t^2 U_t &= \frac{-2i\gamma_-^2 - 8i|\gamma_-|^4}{1 - 16|\gamma_-|^4} \delta_+(0) U_t^* U_t + \alpha^2 U_t^* U_t b_t^2 \\ &\quad -4i\gamma_- \alpha^2 b_t^\dagger U_t^* U_t b_t - 4\gamma_-^2 \alpha^2 b_t^{\dagger 2} U_t^* U_t \end{aligned} \tag{5.4}$$

PROOF. For the proof we first compute

$$U_t^* b_t^2 U_t = U_t^* [b_t^2, U_t] + U_t^* U_t b_t^2$$

We know from (4.8) that

$$[b_t^2, U_t] = -i4\gamma_- b_t^\dagger U_t b_t - 2i\gamma_-^2 \delta_+(0) U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t.$$

Hence

$$\begin{aligned} U_t^* b_t^2 U_t &= U_t^* (-i4\gamma_- b_t^\dagger U_t b_t - 2i\gamma_-^2 \delta_+(0) U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t) + U_t^* U_t b_t^2 \\ &= -4i\gamma_- U_t^* b_t^\dagger U_t b_t - 2i\gamma_-^2 \delta_+(0) U_t^* U_t - 4\gamma_-^2 U_t^* b_t^{\dagger 2} U_t + U_t^* U_t b_t^2 \\ &= U_t^* U_t b_t^2 - 2i\gamma_-^2 \delta_+(0) U_t^* U_t - 4i\gamma_- U_t^* b_t^\dagger U_t b_t - 4\gamma_-^2 U_t^* b_t^{\dagger 2} U_t. \end{aligned} \tag{5.5}$$

It follows from (5.2) that

$$-4i\gamma_- U_t^* b_t^\dagger U_t b_t = \frac{-4i\gamma_-}{1 - 4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{8|\gamma_-|^2}{1 - 4|\gamma_-|^2} U_t^* U_t b_t^2.$$

On the other hand,

$$\begin{aligned} &-4\gamma_-^2 U_t^* b_t^{\dagger 2} U_t \\ &= -4\gamma_-^2 ([U_t^*, b_t^{\dagger 2}] + b_t^{\dagger 2} U_t^*) U_t \\ &= -4\gamma_-^2 (i4\overline{\gamma}_- b_t^\dagger U_t^* b_t + 2i\overline{\gamma}_-^2 \delta_+(0) U_t^* - 4\overline{\gamma}_-^2 U_t^* b_t^2 + b_t^{\dagger 2} U_t^*) U_t \\ &= -4\gamma_-^2 (i4\overline{\gamma}_- b_t^\dagger U_t^* b_t + 2i\overline{\gamma}_-^2 \delta_+(0) U_t^* - 4\overline{\gamma}_-^2 U_t^* b_t^2 + b_t^{\dagger 2} U_t^*) U_t \\ &= -16i|\gamma_-|^2 \gamma_- b_t^\dagger U_t^* b_t U_t - 8i|\gamma_-|^4 \delta_+(0) U_t^* U_t + 16|\gamma_-|^4 U_t^* b_t^2 U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t^* U_t. \end{aligned}$$

Then (5.5) becomes

$$\begin{aligned} U_t^* b_t^2 U_t &= U_t^* U_t b_t^2 - 2i\gamma_-^2 \delta_+(0) U_t^* U_t \\ &\quad + \frac{-4i\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{8|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 \\ &\quad -16i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t U_t - 8i|\gamma_-|^4 \delta_+(0) U_t^* U_t + 16|\gamma_-|^4 U_t^* b_t^2 U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t^* U_t. \end{aligned}$$

Hence

$$\begin{aligned} (1-16|\gamma_-|^4) U_t^* b_t^2 U_t &= U_t^* U_t b_t^2 - 2i\gamma_-^2 \delta_+(0) U_t^* U_t \\ &\quad + \frac{-4i\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{8|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 \\ &\quad -16i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t U_t - 8i|\gamma_-|^4 \delta_+(0) U_t^* U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t^* U_t \\ &= \frac{1+4|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 + (-2i\gamma_-^2 - 8i|\gamma_-|^4)\delta_+(0) U_t^* U_t \\ &\quad + \frac{-4i\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t - 16i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t^* U_t. \end{aligned}$$

In view of (5.3) one obtains

$$-16i|\gamma_-|^2\gamma_- b_t^\dagger U_t^* b_t U_t = \frac{-16i|\gamma_-|^2\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t - \frac{32|\gamma_-|^2\gamma_-^2}{1-4|\gamma_-|^2} b_t^{\dagger 2} U_t^* U_t.$$

Hence

$$\begin{aligned} &(1-16|\gamma_-|^4) U_t^* b_t^2 U_t = \\ &= \frac{1+4|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 + (-2i\gamma_-^2 - 8i|\gamma_-|^4)\delta_+(0) U_t^* U_t + \frac{-4i\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t \\ &\quad + \frac{-16i|\gamma_-|^2\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t - \frac{32|\gamma_-|^2\gamma_-^2}{1-4|\gamma_-|^2} b_t^{\dagger 2} U_t^* U_t - 4\gamma_-^2 b_t^{\dagger 2} U_t^* U_t \\ &= \frac{1+4|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 + (-2i\gamma_-^2 - 8i|\gamma_-|^4)\delta_+(0) U_t^* U_t \\ &\quad + \frac{-4i\gamma_- - 16i|\gamma_-|^2\gamma_-}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{-32|\gamma_-|^2\gamma_-^2 - 4\gamma_-^2(1-4|\gamma_-|^2)}{1-4|\gamma_-|^2} b_t^{\dagger 2} U_t^* U_t \\ &= \frac{1+4|\gamma_-|^2}{1-4|\gamma_-|^2} U_t^* U_t b_t^2 + (-2i\gamma_-^2 - 8i|\gamma_-|^4)\delta_+(0) U_t^* U_t \\ &\quad + \frac{-4i\gamma_-(1+4|\gamma_-|^2)}{1-4|\gamma_-|^2} b_t^\dagger U_t^* U_t b_t + \frac{-4\gamma_-^2(1+4|\gamma_-|^2)}{1-4|\gamma_-|^2} b_t^{\dagger 2} U_t^* U_t, \end{aligned}$$

which completes the proof. qed

Lemma 5.3 *The equation satisfied by $x_t = U_t^* x U_t$ is*

$$\partial_t x_t = 4\mathrm{Re}(\gamma_-^2)\delta_+(0) x_t - i b^{\dagger 2} x_t + i x_t b_t^2. \tag{5.6}$$

PROOF. Differentiating by the Leibnitz rule equation (5.1), we find

$$\begin{aligned}\partial_t x_t &= i b_t^{\dagger 2} x_t - i x_t b_t^2 \\ &\quad + i(4\gamma_-^2 - 1)U_t^* x b_t^{\dagger 2} U_t + (h.c.) - 4\gamma_- U_t^* x b_t^\dagger U_t b_t + (h.c.) \\ &\quad - 4\mathrm{Re}(\gamma_-^2)\delta_+(0) x_t. \end{aligned} \tag{5.7}$$

The result then follows using Lemmas 5.1 and 5.2. In fact, we compute

$$\begin{aligned}& i(4\gamma_-^2 - 1)U_t^* x b_t^{\dagger 2} U_t + (h.c.) - 4\gamma_- U_t^* x b_t^\dagger U_t b_t + (h.c.) = \\ &= i(4\gamma_-^2 - 1)U_t^* x b_t^{\dagger 2} U_t - i(4\overline{\gamma}_-^2 - 1)U_t^* b_t^2 x^* U_t - 4\gamma_- U_t^* x b_t^\dagger U_t b_t - 4\overline{\gamma}_- b_t^\dagger U_t^* b_t x^* U_t. \end{aligned}$$

With no loss of generality we can omit x from the equation. Inserting the results obtained above we find

$$\begin{aligned}& i(4\gamma_-^2 - 1)U_t^* b_t^{\dagger 2} U_t = \\ &= i(4\gamma_-^2 - 1)\Big(\alpha^2 b_t^{\dagger 2} U_t^* U_t + \frac{2i\overline{\gamma}_-^2 + 8i|\gamma_-|^4}{1 - 16|\gamma_-|^4}\delta_+(0)U_t^* U_t \\ &\quad + 4i\overline{\gamma}_- \alpha^2 b_t^\dagger U_t^* U_t b_t - 4\overline{\gamma}_-^2 \alpha^2 U_t^* U_t b_t^2\Big) \\ &= i(4\gamma_-^2 - 1)\alpha^2 b_t^{\dagger 2} U_t^* U_t + \frac{-2(4\gamma_-^2 - 1)(\overline{\gamma}_-^2 + 4|\gamma_-|^4)}{1 - 16|\gamma_-|^4}\delta_+(0)U_t^* U_t \\ &\quad - 4(4\gamma_-^2 - 1)\overline{\gamma}_- \alpha^2 b_t^\dagger U_t^* U_t b_t - 4i(4\gamma_-^2 - 1)\overline{\gamma}_-^2 \alpha^2 U_t^* U_t b_t^2. \end{aligned}$$

Similarly

$$\begin{aligned}& -i(4\overline{\gamma}_-^2 - 1)U_t^* b_t^2 x^* U_t = \\ &= -i(4\overline{\gamma}_-^2 - 1)\alpha^2 U_t^* U_t b_t^2 + \frac{-2(4\overline{\gamma}_-^2 - 1)(\gamma_-^2 + 4|\gamma_-|^4)}{1 - 16|\gamma_-|^4}\delta_+(0)U_t^* U_t \\ &\quad - 4(4\overline{\gamma}_-^2 - 1)\gamma_- \alpha^2 b_t^\dagger U_t^* U_t b_t + 4i(4\overline{\gamma}_-^2 - 1)\gamma_-^2 \alpha^2 b_t^{\dagger 2} U_t^* U_t. \end{aligned}$$

On the other hand,

$$-4\gamma_- U_t^* x b_t^\dagger U_t b_t = -4\gamma_-(\alpha b_t^\dagger U_t^* U_t b_t + 2i\overline{\gamma}_- \alpha U_t^* U_t b_t^2)$$

and

$$-4\overline{\gamma}_- b_t^\dagger U_t^* x b_t U_t = -4\overline{\gamma}_-(\alpha b_t^\dagger U_t^* U_t b_t - 2i\gamma_- \alpha b_t^{\dagger 2} U_t^* U_t).$$

Summing up the above four expressions one finds

$$\begin{aligned}& \frac{(2 - 32|\gamma_-|^4)(\gamma^2 + \overline{\gamma}_-^2)}{1 - 16|\gamma_-|^4}\delta_+(0)U^* U \\ &\quad + i\{-(1 - 16|\gamma_-|^4)\alpha^2 + 8\alpha|\gamma_-|^2\} b_t^{\dagger 2} U_t^* U_t \\ &\quad + i\{(1 - 16|\gamma_-|^4)\alpha^2 - 8\alpha|\gamma_-|^2\} U_t^* U_t b_t^2 \\ &\quad + \{4\alpha^2(\gamma + \overline{\gamma}) - 16\alpha^2|\gamma_-|^2(\gamma_- + \overline{\gamma}_-) - 4\alpha(\gamma_- + \overline{\gamma}_-)\} b_t^\dagger U_t^* U_t b_t \\ &= 4\mathrm{Re}(\gamma_-^2)\delta_+(0)U_t^* U_t - i b^{\dagger 2} U_t^* U_t + i U_t^* U_t b_t^2. \end{aligned}$$

This completes the proof. qed

6 Renormalization and formal unitarity of the solution

Starting with

$$\partial_t U_t = -i(b_t^{\dagger 2} + b_t^2)U_t, \qquad U_0 = I, \tag{6.1}$$

we have found the normal form:

$$\partial_t U_t = i(4\gamma_-^2 - 1)b_t^{\dagger 2}U_t - 4\gamma_- b_t^\dagger U_t b_t - iU_t b_t^2 - 2\gamma_-^2\delta_+(0)U_t, \tag{6.2}$$

see Proposition 4.3. Note that (6.2) involves the ill-defined quantity $\gamma_-^2\delta_+(0)U_t$ where γ_- is a complex number. However, the addition of a quantity of the form $\rho_0 = \rho\delta_+(0)$, where ρ is a real constant, to the Hamiltonian in (6.1) does not change the basic quantity of interest, which is not the operator U_t itself but the flow $U_t^* x U_t$. After this addition we obtain

$$\partial_t U_t = -i(b_t^{\dagger 2} + b_t^2 + \rho\delta_+(0))U_t, \qquad U_0 = I, \tag{6.3}$$

and its normal form:

$$\partial_t U_t = i(4\gamma_-^2 - 1)b_t^{\dagger 2}U_t - 4\gamma_- b_t^\dagger U_t b_t - iU_t b_t^2 - 4(\mathrm{Re}(\gamma_-^2) + i\{\mathrm{Im}(\gamma_-^2) - \rho/4\})\delta_+(0)U_t.$$

Therefore, if $\mathrm{Re}(\gamma_-^2) = 0$, then by putting $\rho = 4\,\mathrm{Im}(\gamma_-^2)$ we see that the ill-defined quantity vanishes identically.

On the other hand, by a direct computation we obtain

Lemma 6.1 *The right hand side of equation (5.1) is identically zero for any choice of* γ_-.

The above result shows that the normal form equation (6.2) satisfies the *formal unitarity* condition. Therefore the right hand side of (5.7) is identically zero as we expected. Notice that the same condition ($\mathrm{Re}(\gamma_-^2) = 0$) which renormalizes the equation for U also renormalizes the equation for the flow j_t.

Our next task is to investigate the regularity and unitarity condition for U_t or j_t obeying the renormalized equation. The research is now in progress.

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