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EXTENDED FORMAL POWER SERIES AND G-FUNCTIONS

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At first, let us consider a formal power series ring $R = k[[x]]$ where k is a field. The fraction field of R is $\mathbb{Q}(R) = k((x))$. Every element of $k((x))$ is expressed as a power series with finite negative exponents. But when we consider a power series ring of several indereminates $R = k[[x_1, ..., x_n]]$, some elements of $\mathbb{Q}(R)$ can not be expressd as a power series. For example, consider

$$\frac{1}{x+y} \in \mathbb{Q}(k[[x, y]]).$$

Sometimes, we want to express every element of $\mathbb{Q}(R)$ as a formal power with possibly negative exponents. So I introduced extended formal power series rings. [5]

Let $\alpha = (\alpha_1, ..., \alpha_m)$ be a vector in $\mathbb{R}^m$, and let $\underline{n} = (n_1, ..., n_m)$ be an integer vector in $\mathbb{Z}^m$. Fixing $\underline{\alpha}$, $L = L(\underline{n})$ denotes the linear form

$$\underline{\alpha} \cdot \underline{n} = \alpha_1 n_1 + ... + \alpha_m n_m.$$

We abbreviate $\sum a(\underline{i})\underline{x}^{\underline{i}}$ for

$$\sum_{i_1=-\infty}^{\infty} ... \sum_{i_m=-\infty}^{\infty} a_{i_1...i_m} {x_1}^{i_1} ... x_m^{i_m}.$$

The following definitions are essential.

Definition 1. A subset $I \subset \mathbb{Z}^m$ is L-finite iff $\forall N \in \mathbb{Z}$

$$\#(I \cap \{\underline{n} | L(\underline{n}) < N\}) < \infty$$

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Definition 1'. $f = \sum a(\underline{i})\underline{x}^{\underline{i}}$ is L-finite iff $I = \{\underline{i} | a(\underline{i}) \neq 0\}$ is L-finite.

Definition 2. $K_L = k((\underline{x}))_L = k((x_1, ..., x_m)) = \{L - finite\ series\}$.

Under these definitons, we have the following:

Theorem 0. (1) $k((\underline{x}))_L$ is a $k[\underline{x}]$-algebra. (2) If $\alpha_1, ..., \alpha_m$ are linearly independent over $\mathbb{Q}$ then $K = K_L$ is a field containing $k(\underline{x})$.

Remark. When $char(k) > 0$ many results are obtained. In this note we restrict ourselves to relation to G-functions.

From now on let k be a number field and Σ be the set of all places of k, and $|.|_v$ be the normalized absolute value corresponding to $v \in \Sigma$. Let $f = \sum_{n=0}^{\infty} a_n x^n \in k[[x]]$. The definition of the G-function is the following.

f is an G-function iff

(1) $\sigma(f) < \infty$

(2) f $is\ D-finite.$

Here $\sigma(f) = \overline{lim}_{n\to\infty}\frac{1}{n}\sum_v Max_{m\leq n}(log^+|a_m|_v)$, and D-finite means that f satisfies a linear differential equation over $k(x)$. It is well known that this definiton is equivalent to the Siegel's original definition. Further we may take f from $k((x))$.

By using our "extended power series" we can define G-functions of several vaiables naturally. That is : f is an "extended" G-function iff

(1) $f \in K_L,\ \sigma(f) = \overline{lim}_{N\to\infty}\frac{1}{N}\sum_v Max_{L(\underline{n})<N}(log^+(|a_n|_v) < \infty.$

(2) $f \in K_L$ is D-finite (f is contained in a $\frac{d}{dx_i}$-stable $k(\underline{x})$-vectore subseace $V \subset K_L$).

Next we consider the diagonal maps. For

$$f = \sum_{n_1=0}^{\infty}\sum_{n_m=0}^{\infty} a_{n_1,\ldots,n_m}x_1^{n_1}\ldots x_m^{n_m} \in k[[x_1,\ldots,x_m]],$$

diagonal map I is defined as

$$I(f) = \sum_{n=0}^{\infty} a_{n,\ldots,n}t^n \in k[[t]].$$

It is easy to see that the diagonal map I is defined for "extended formal power series rings" K_L.

It can be proved that if $f \in K_L$ is D-finite then $I(f) \in K((t))$ is also D-finite . So we have that

$$f \in K_L : \text{"}extended\text{"}G - function \Rightarrow I(f) : G - function.$$

Recall the following conjecture of Christol:

Every globlly bounded G-function is a diagonal of some rational function. Here "globally bounded" for series $f = \sum a_n x^n$ means that coefficients $a_n \in \mathbb{O}[\frac{1}{N}]$ for every n, where $\mathbb{O}$ is the ring of integers in the number field k and N is a natural integer. In this conjecture, the rational function means an elements in $K[\underline{x}]_{(x)}$. But in our situation we can take elements from $k(\underline{x})$.

It is sometimes possible to prove an "extended" G-function to be a rational function. The method of Gelgond, Chudnovskys are available for elements in $k((x_1,\ldots,x_\nu))_L$. The following is the analogy for the Chudnovskys criterion for rationality for elements in $k[[x_1,\ldots,x_\nu]]$.

Proposition. Let $Y = (y_0,\ldots,y_{\mu-1}) \subset K((x_1,\ldots,x_\nu))_L$, let $\tau > 0$, and let $V \subset \Sigma$ be some subset of places of k. Assume that for each $v \in V$ the $y_i's$ converge on a polydisk $|x_i|_v < \kappa_{i,v}$ $(i = 1,\ldots,\nu)$. If the following inequality holds

$$(*)\ \sigma_{notV}(Y) + \tau\sigma(Y) < \sum_{v\in V}[1 - (\frac{1}{\mu}(1+\frac{1}{\tau}))^{\frac{1}{\nu}}]\cdot(\sum_{i=1}^{\nu} log\kappa_{i,v}),$$

then y_i's are linearly dependent over $k(\underline{\xi})$ where $\underline{\xi} = (\xi_1, ..., \xi_\nu)$, $\xi_i = x_i^{\frac{1}{n}}$ for some $n > 0$. It is a question to prove that $y_i's$ are linearly dependent over $k(\underline{x})$.

References

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