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A GENERALIZATION OF THE SIZES OF DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS TO G-FUNCTION THEORY

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This is a summary about "a generalization of the sizes of differential equations and its applications to G-function theory" [5].

Let K be an algebraic number field of a finite degree. We consider a linear differential equation:

$$\frac{d}{dx}y = Ay, \quad (A \in M_n(K(x))). \tag{1}$$

Let us define the sizes and the global radii regarding differential equation (1).

For a place v of K we put

$$\begin{cases} |p|_v := p^{\frac{-d_v}{d}} & \text{if } v \mid p \quad (p : \text{prime}), \\ |\xi|_v := |\xi|^{\frac{d_v}{d}} & \text{if } v \mid \infty \quad (\xi \in K), \end{cases}$$

where $d = [K : \mathbb{Q}]$ and $d_v = [K_v : \mathbb{Q}_p]$.

We define a pseudo valuation on $M_{n_1,n_2}(K)$: for $M = (m_{i,j})_{j=1,\dots,n_2}^{i=1,\dots,n_1} \in M_{n_1,n_2}(K)$,

$$|M|_v := \max_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_2}} |m_{i,j}|_v.$$

For $Y_i \in M_{n_1,n_2}(K)$, we consider the Laurent series $Y = \sum_{i=-N}^{\infty} Y_i x^i \in M_{n_1,n_2}(K((x)))$ with $N \in \mathbb{N} \cup \{0\}$.

We write $\log^+ a := \log\max(1, a)$ $(a \in \mathbb{R})$. André's symbol $h_{\cdot,\cdot}(\cdot)$ in [1] is defined by

$$h_{v,0}(Y) := \max_{i \leq 0} \log^+ |Y_i|_v,$$

$$h_{v,m}(Y) := \frac{1}{m} \max_{i \leq m} \log^+ |Y_i|_v \quad (m \neq 0).$$

Definition 2. (Cf. [1]) We define *the size* of $Y \in M_{n_1,n_2}(K((x)))$ as

$$\sigma(Y) := \overline{\lim_{m\to\infty}} \sum_v h_{v,m}(Y)$$

and *the global radii* of Y as

$$\rho(Y) := \sum_v \overline{\lim_{m\to\infty}}\, h_{v,m}(Y),$$

where $\sum_v$ means that v ranges over all places of K.

The following definition coincides with the one in [6] in the case of $Y \in K[[x]]$.

Definition 3. We call $Y \in M_{n_1,n_2}(K((x)))$ with $\sigma(Y) < \infty$ *a matrix of G-functions.*

For $f = f(x) = \sum_{i=0}^{N} f_i x^i \in K[x]$ and for every place v of K, the Gauss absolute value is defined by $|f|_v := \max_{i=0,\dots,N} |f_i|_v$.

For every place v with $v \nmid \infty$ and for $f, g \in K[x]$ with $g \not\equiv 0$, the Gauss absolute value is extended to $K(x)$ by

$$|\frac{f}{g}|_v := \frac{|f|_v}{|g|_v}.$$

We also define a pseudo valuation on $M_n(K(x))$: for $M = (m_{i,j})_{i,j=1,\dots,n} \in M_n(K(x))$,

$$|M|_v := \max_{i,j=1,\dots,n} |m_{i,j}|_v.$$

Suppose that $A \in M_n(K(x))$. A sequence $\{E_i\}_{i=0,1,\dots} \subset M_n(K(x))$ is defined by

$$E_0 := I$$

and recursively for $i = 1, 2, \dots,$

$$E_{i+1} := \frac{1}{i+1}\left(\frac{d}{dx}E_i + E_i A\right).$$

For this sequence $\{E_i\}_{i=0,1,\dots} \subset M_n(K(x))$ and for every place $v \nmid \infty$, we put

$$h_{v,0}(\{E_i\}) := \log^+ |E_0|_v,$$
$$h_{v,m}(\{E_i\}) := \frac{1}{m} \max_{i \le m} \log^+ |E_i|_v \quad (m = 1, 2, \dots).$$

Definition 4. We define *the size* of A as

$$\sigma(A) := \overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} h_{v,m}(\{E_i\})$$

and *the global radii* of A as

$$\rho(A) := \sum_{v\nmid\infty} \overline{\lim_{m\to\infty}}\, h_{v,m}(\{E_i\}),$$

where $\sum_{v\nmid\infty}$ means that v ranges over all finite places of K.

Definition 5. We call $\frac{d}{dx} - A$ with $\sigma(A) < \infty$ *G-operator* and $\frac{d}{dx} - A$ with $\rho(A) < \infty$ *the Arithmetic type.*

According to these notations, we state known results:

Theorem 6. *(Cf.* [1], [2], [3]*) Suppose that* $A \in M_n(K(x))$ *and suppose that* A *has at most the simple pole at* $x = 0$. *For a solution,* y, *of differential equation* (1), *let* y *belong to* $K[[x]]$ *and its entries be linear independent over* $K(x)$. *Then the following five assertions are equivalent:*

$$\sigma(y) < \infty, \tag{6.1}$$

$$\sigma(A) < \infty, \tag{6.2}$$

$$\sigma(A^*) < \infty, \tag{6.3}$$

$$\rho(A) < \infty, \tag{6.4}$$

$$\rho(A^*) < \infty \tag{6.5}$$

where $A^* = -{}^tA$. *Moreover they imply*

$$\rho(y) < \infty. \tag{6.6}$$

Theorem 6 is the main theorem in [1]. Before stating André results, we need a definition.
After a transformation of differential equation (1), there exists the unique matrix solution of differential equation (1), Yx^C with $Y \in Gl_n(K[[x]]), Y_{|x=0} = I$, where C is the residue of A at $x = 0$. This $Y \in Gl_n(K[[x]])$ is called *the normalized uniform part of the solution* of differential equation (1).
He proved Theorem 6 by using the following:

Theorem 7. *(Cf.* [1]*) Suppose that* $A \in M_n(K(x))$ *and suppose that* A *has at most the simple pole at* $x = 0$. *let* $Y \in Gl_n(K[[x]])$ *be the normalized uniform part of differential equation* (1). *Let differential equation* (1) *be Fuchsian and let all eigenvalues of the residue matrix of* A *at* $x = 0$ *be rational numbers. Then*

$$\sigma(A) < \infty \text{ if and only if } \rho(A) < \infty, \tag{7.1}$$

$$\rho(A) < \infty \text{ implies } \rho(Y) < \infty, \tag{7.2}$$

$$\rho(Y) < \infty \text{ implies } \sigma(Y) < \infty. \tag{7.3}$$

i.e.,

$$\sigma(A) < \infty \text{ implies } \sigma(Y) < \infty.$$

Now for a differential equation

$$\frac{d}{dx}X = AX - XB, \quad (A, B \in M_n(K(x))), \tag{8}$$

we introduce its new size $\sigma(A,B)$ of differential equation (8).

Let us define another sequence $\{F_i\}_{i=0,1,\dots} \subset M_n(K(x))$ as

$$F_0 := I$$

and recursively for $i = 1, 2, \dots$,

$$F_{i+1} := \frac{1}{i+1}\left(\frac{d}{dx}F_i - AF_i + F_iB\right).$$

Definition 9. We define *the size* of A and B as

$$\sigma(A,B) := \overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} h_{v,m}(\{F_i\})$$

and *the global radii* of A and B as

$$\rho(A,B) := \sum_{v\nmid\infty} \overline{\lim_{m\to\infty}}\, h_{v,m}(\{F_i\}).$$

Namely $\sigma(A) = \sigma(0,A)$.

This size $\sigma(A,B)$ has the following properties:

Theorem 10. *(Cf. [5]) For any $A, B, C \in M_n(K(x))$ and any $T \in Gl_n(K(x))$, the followings hold:*

(10.1) $$\sigma(A,A) = 0,$$

(10.2) $$\sigma(A,B) = \sigma(T[A], T[B]),$$

(10.3) $$\sigma(A,B) \le \sigma(A,C) + \sigma(C,B).$$

Here $T[A] = TAT^{-1} + (\frac{d}{dx}T)T^{-1}$.

An application of Theorem 10 as the converse proposition of Theorem 7 is following:

Theorem 11. *(Cf. [5]) Let $A \in M_n(K(x))$ and let Y be the normalized uniform part of the solution of differential equation (1). Let $u \in \mathcal{O}_K[x]$ be a common denominator of A, where $\mathcal{O}_K$ denotes the integer ring of K. Let $s := \max(\deg u, \deg(uA))$. Suppose that*

$$\mathcal{E} := \{\text{Eigenvalues of the residue of } A\ \} \subset \mathbb{Q}.$$

Then

(11.1) $$\sigma(A) \le 9n^4(s+1)\sigma(Y) + 3\log N_{\mathcal{E}} + 3 \sum_{\substack{p \mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + (s+1)h_\infty(u) + \log(s+1) + 3(n-1),$$

where $h_\infty(u) := \frac{1}{m+1}\sum_{v\mid\infty} \max_{i\le m} \log^+ |u_i|_v$ and $N_{\mathcal{E}} \in \mathbb{N}$ is a common denominator of $\mathcal{E}$. i.e.,

$$\sigma(Y) < \infty \text{ implies } \sigma(A) < \infty.$$

Remark 12. The same result on the finiteness by another method was published [4].

From Theorem 7, Theorem 11 and the uniqueness of the normalized uniform part, we summarize them as follows:

Theorem 13. *Under the assumptions of Theorem 7, the following eight assertions are equivalent:*

$$\sigma(Y) < \infty, \tag{13.1}$$
$$\sigma(A) < \infty, \tag{13.2}$$
$$\rho(Y) < \infty, \tag{13.3}$$
$$\rho(A) < \infty, \tag{13.4}$$
$$\sigma(Y^{-1}) < \infty, \tag{13.5}$$
$$\sigma(A^*) < \infty, \tag{13.6}$$
$$\rho(Y^{-1}) < \infty, \tag{13.7}$$
$$\rho(A^*) < \infty, \tag{13.8}$$

where $A^* = -\,{}^tA$. *More precisely*

$$\sigma(A) = \sigma(A^*), \tag{13.9}$$
$$\rho(A) = \rho(A^*). \tag{13.10}$$

Remark 14. Equation (13.10) is derived using a different method in [1].

References

[1] Y. André, *G-functions and Geometry*, Max-Planck-Institut., Bonn, 1989.

[2] E. Bombieri, *On G-functions*, Recent progress in analytic number theory **2** (1981), Academic Press, New York, 1 – 67.

[3] D. V. Chudnovsky, G. V. Chudnovsky, *Applications of Padé approximations to diophantine inequalities in values of G-functions*, Lect. Notes in Math. **1135** (1985), Springer-Verlag, Berlin, Heidelberg, New York, 9 – 51.

[4] B. Dwork, G. Gerotto, F. J. Sullivan, *An Introduction to G-functions*, Annals of Math. Studies **133** (1994), Princeton University Press, Princeton, New Jersey.

[5] M. Nagata, *A generalization of the sizes of differential equations and its applications to G-function theory* (1994), Preprint series in Math. (Tokyo Inst. Tech.).

[6] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., Phys. Math. Kl. nr.1 (1929).

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