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Sum of Kloosterman zeta functions

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Kloosterman zeta function is closely connected with Linnik - Selberg conjecture or Selberg's first eigenvalue conjecture (cf.Goldfeld-Sarnak [2]), but its properties are not known much yet. In this paper, I want to show that a certain series of Kloosterman zeta functions weighted by the Fourier coefficients of Maass wave form can be expressed by the integral of the functions which we studied in [7]. We will use the following notations.

Kloosterman sum $S(n,m;c) := \sum_{\substack{d \bmod (c) \\ dd' \equiv 1 \bmod (c)}}{}' \exp(2\pi i \frac{nd+md'}{c})$.

The prime $'$ shows that d runs over the integers such that $(c,d)=1$. Especially $S(0,m;c)$ is the so-called Ramanujan sum.

Kloosterman zeta function $Z_{n,m}(s) := \sum_{c=1}^{\infty} S(n,m;c)c^{-2s}$.

According to Weil's estimate on Kloosterman sum $S(n,m;c) \ll c^{1/2}d(c)$, the series defining $Z_{n,m}(s)$ converges absolutely for $\mathrm{Re}s > 3/4$.

$\hat{\Gamma}(s;\mu,\nu) = \Gamma(\frac{s+\mu+\nu}{2})\Gamma(\frac{s+\mu-\nu}{2})\Gamma(\frac{s-\mu+\nu}{2})\Gamma(\frac{s-\mu-\nu}{2})$.

$G(\alpha,\beta;\gamma;z) = F(\alpha,\beta;\gamma;z) - 1$.

Here $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function. We have

$$G(\alpha,\beta;\gamma;z) = \frac{\alpha\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} z \int_0^1 \int_0^1 t^{\beta}(1-t)^{\gamma-\beta-1}(1-txz)^{-\alpha-1}dtdx$$

for $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > -1$.

§1. Let $f(z)$ be a Maass wave form with respect to $SL_2(\mathbf{Z})$. Namely, $f(z)$ is a non-constant function on the upper half plane $\mathcal{H}$, belongs to $L_2(SL_2(\mathbf{Z})\backslash\mathcal{H})$ and is an eigenfunction of each n-th Hecke operator $T_n, (n=1,2,\ldots)$ and a non-Euclidean Laplacian $\Delta = -y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$. When the eigenvalue of Δ is $1/4+\kappa^2$, then $f(z)$ has the Fourier series expansion

$$f(z)=\sum_{n\neq 0}\rho(n)y^{1/2}K_{i\kappa}(2\pi|n|y)e^{2\pi inx}.$$

Let us suppose that $\rho(-n)=\varepsilon_f\rho(n)$, $\varepsilon_f=\pm 1$. The value ε_f is called the parity of $f(z)$. We further assume that $\rho(n)=O(n^{\eta_0})$ for some $\eta_0>0$. Up to now, it is known that $\eta_0\leq 5/28$.(cf. [1] Bump et al.)

We consider the Dirichlet series

$$\mathcal{D}(s,r)=\sum_{n=1}^{\infty}\frac{\rho(n)e^{2\pi irn}}{n^s},\qquad (r\in\mathbf{R}).$$

Lemma 1 . *Let c,d be relatively prime integers and $dd'\equiv 1 \pmod c$. Then $\mathcal{D}(s,d/c)$ can be continued to the whole s plane as a holomorphic function and satisfies the following functional equation:*

$$\mathcal{D}(s,d/c)=c^{1-2s}\Omega_{i\kappa}(s)\{\cos(\pi s)\mathcal{D}(1-s,-d'/c)-\varepsilon_f\cos(\pi i\kappa)\mathcal{D}(1-s,d'/c)\},\tag{1}$$

where the Γ-factor is defined by

$$\Omega_{i\kappa}(s)=-\frac{(2\pi)^{2s-1}\pi}{\Gamma(s+i\kappa)\Gamma(s-i\kappa)\sin\pi(s+i\kappa)\sin\pi(s-i\kappa)}.\tag{2}$$

Remark. I found recently that this functional equation has already been obtained by Kuznetsov [4] or Meurman [5], but I will write the outlines of proof to make sure the points.

Proof. Let $\tilde{f}(z)=\frac{\partial}{\partial x}f(z)$, $(z=x+iy)$. For $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbf{Z})$, $(c\neq 0)$, we have the automorphic property of $\tilde{f}(z)$:

$$-\frac{1}{c^2y^2}\tilde{f}(\frac{a}{c}+\frac{i}{c^2y})=\tilde{f}(-\frac{d}{c}+iy).\tag{3}$$

We define the (translations of) Mellin transforms of $f(z)$ and $\tilde{f}(z)$ by

$$L(s,x)=\int_0^\infty f(x+iy)y^{s-1/2}\frac{dy}{y}$$

and

$$\tilde{L}(s,x)=\int_0^\infty \tilde{f}(x+iy)y^{s+1/2}\frac{dy}{y}.$$

First let $f(z)$ be an even cusp form, i.e. $\varepsilon_f = 1$. By the Fourier expansion of $f(z)$, we have

$$L(s,x) = 2^{-1}\pi^{-s}\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})\sum_{n=1}^{\infty}\frac{\rho(n)\cos(2\pi nx)}{n^s},$$

$$\tilde{L}(s,x) = -\pi^{-s}\Gamma(\frac{s+1+i\kappa}{2})\Gamma(\frac{s+1-i\kappa}{2})\sum_{n=1}^{\infty}\frac{\rho(n)\sin(2\pi nx)}{n^s}.$$

Therefore,

$$\mathcal{D}(s,x) = \frac{2\pi^s}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})}L(s,x) - \frac{i\pi^s}{\Gamma(\frac{s+1+i\kappa}{2})\Gamma(\frac{s+1-i\kappa}{2})}\tilde{L}(s,x).$$

On the other hand, the automorphic properties of $f(z)$ and $\tilde{f}(z)$ yield the following functional equations:

$$L(s,\frac{d}{c}) = c^{1-2s}L(1-s,-\frac{a}{c}),$$

$$\tilde{L}(s,\frac{d}{c}) = -c^{1-2s}\tilde{L}(1-s,-\frac{a}{c}).$$

The assertion (1) can be obtained by combining these equations.

The case that $f(z)$ is odd (i.e. $\varepsilon_f = -1$) can be treated similarly.

Let $\sigma_1 > 1, \sigma_2 < -\eta_0$ be real numbers, and let $\omega(x)$ be a test function defined on $[0,\infty)$. We put

$$\hat{\omega}(s) = \int_0^{\infty}\omega(x)x^{s-1}dx,$$

the Mellin transform of $\omega(x)$. We assume that the above integral converges absolutely for $\mathrm{Re}(s) > 1$, and the function $\hat{\omega}(s)$ can be continued holomorphically to the domain containing $\sigma_2 \le \mathrm{Re}(s)$. Furthermore, we assume that it has a sufficient rapid decay when $|\mathrm{Im}(s)| \to \infty$. This condition will be used later only when we move the line of integration. I do not set here how rapidly $\hat{\omega}(s)$ should decay. We have, by the Mellin inversion formula, that

$$\omega(x) = \frac{1}{2\pi i}\int_{(\sigma_1)}\hat{\omega}(s)x^{-s}ds \tag{4}$$

where the contour of integration (σ_1) is the straight line from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$.

First we consider the sum

$$\sum_{n=1}^{\infty}\rho(n)S(n,m;c)\omega(n). \tag{5}$$

The test function $\omega(x)$ should be taken so that the above series converges absolutely.

Lemma 2 . *As to the sum (5), we have*

$$\sum_{n=1}^{\infty}\rho(n)S(n,m;c)\omega(n)$$
$$=\frac{1}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)c^{1-2s}\cos(\pi s)\Big\{\sum_{n=1}^{\infty}\rho(n)S(0,m-n;c)n^{s-1}\Big\}ds$$
$$-\frac{\varepsilon_f\cos(\pi i\kappa)}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)c^{1-2s}\Big\{\sum_{n=1}^{\infty}\rho(n)S(0,m+n;c)n^{s-1}\Big\}ds$$

Proof. We replace $\omega(n)$ in (5) with (4),then (5) is transformed to

$$\frac{1}{2\pi i}\int_{(\sigma_1)}\hat{\omega}(s)\Big(\sum_{n=1}^{\infty}\rho(n)S(n,m;c)n^{-s}\Big)ds$$
$$=\sum_{d\,\mathrm{mod}(c)}{}' e^{2\pi i\frac{md'}{c}}\frac{1}{2\pi i}\int_{(\sigma_1)}\hat{\omega}(s)\Big(\sum_{n=1}^{\infty}\rho(n)e^{2\pi i\frac{nd}{c}}n^{-s}\Big)ds$$
$$=\sum_{d\,\mathrm{mod}(c)}{}' e^{2\pi i\frac{md'}{c}}\frac{1}{2\pi i}\int_{(\sigma_1)}\hat{\omega}(s)\mathcal{D}(s,d/c)ds$$

In the last integral, we move the line of integration to $\mathrm{Re}(s)=\sigma_2$, and substitute the functional equation (1) for $\mathcal{D}(s,d/c)$. Since $\mathrm{Re}(s)=\sigma_2<-\eta_0$, the series

$$\mathcal{D}(1-s,\pm d'/c)=\sum_{n=1}^{\infty}\rho(n)e^{\pm 2\pi i\frac{nd'}{c}}n^{s-1}$$

is absolutely convergent. After changing the order of integration and summation, we get the desired equality.

Theorem 1 . *Let s_0 be a complex number with* $\mathrm{Re}(s_0)\gg 0$,*then we have*

$$\sum_{n=1}^{\infty}\rho(n)Z_{n,m}(s_0)\omega(n)$$
$$=\frac{1}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)\cos(\pi s)\Big\{\sum_{n=1}^{\infty}\rho(n)\sigma_{2(1-s-s_0)}(|n-m|)n^{s-1}\Big\}\zeta(2s+2s_0-1)^{-1}ds$$
$$-\frac{\varepsilon_f\cos(\pi i\kappa)}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)\Big\{\sum_{n=1}^{\infty}\rho(n)\sigma_{2(1-s-s_0)}(n+m)n^{s-1}\Big\}\zeta(2s+2s_0-1)^{-1}ds,\quad (6)$$

where we put $\sigma_\alpha(m)=\sum_{d|m}d^\alpha$ $(m\neq 0)$、$\sigma_\alpha(0)=\zeta(-\alpha)$.

Proof. Let s_0 be a complex number with $\mathrm{Re}(s_0) \gg 0$. Multiply c^{-2s_0} on both sides of Lemma 2 and take the summation with respect to c, then we have

$$\sum_{n=1}^{\infty} \rho(n) Z_{n,m}(s_0)\omega(n)$$
$$= \frac{1}{2\pi i}\int_{(\sigma_2)} \hat{\omega}(s)\Omega_{i\kappa}(s)\cos(\pi s)\sum_{n=1}^{\infty}\rho(n)\Big(\sum_{c=1}^{\infty}\frac{S(0,n-m;c)}{c^{2s+2s_0-1}}\Big)n^{s-1}ds$$
$$-\frac{\varepsilon_f\cos(\pi i\kappa)}{2\pi i}\int_{(\sigma_2)} \hat{\omega}(s)\Omega_{i\kappa}(s)\sum_{n=1}^{\infty}\rho(n)\Big(\sum_{c=1}^{\infty}\frac{S(0,n+m;c)}{c^{2s+2s_0-1}}\Big)n^{s-1}ds.$$

The following formulas on Ramanujan sums are well known.

$$\sum_{c=1}^{\infty}\frac{S(0,m;c)}{c^s} = \frac{\sigma_{1-s}(|m|)}{\zeta(s)} \quad (m \neq 0, \mathrm{Re}s > 1),$$

$$\sum_{c=1}^{\infty}\frac{S(0,0;c)}{c^s} = \frac{\varphi(c)}{c^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\mathrm{Re}s > 2).$$

Hence, we get the theorem.

Remark. The above method which transforms the sum of Kloosterman sums to the sum of Ramanujan sums by the functional equation of certain Dirichlet series was used in Motohashi [6].

For a natural number k, we define

$$D_k(s;\alpha,f) = \sum_{m=1}^{\infty}\frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^s} + \varepsilon_f\sum_{\substack{m=1\\ m\neq k}}^{\infty}\frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^s}.$$

(cf.[7]) This Dirichlet series is absolutely convergent for $\mathrm{Re}(s) > 1+\eta_0$ if $\mathrm{Re}(\alpha) \leq 1/2$, and for $\mathrm{Re}(s) > 2\mathrm{Re}(\alpha)+\eta_0$ if $\mathrm{Re}(\alpha) > 1/2$.

Corollary 1 . *Let* $\mathrm{Re}(s_0) \gg 0$, *then we have*

$$\sum_{n=1}^{\infty}\rho(n)\big[Z_{n,k}(s_0)+\varepsilon_f Z_{n,-k}(s_0)\big]\omega(n)$$
$$= \frac{\varepsilon_f}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)\zeta(2s+2s_0-1)^{-1}$$
$$\times\Big\{D_k(1-s;3/2-s-s_0,f)+\varepsilon_f\rho(k)k^{s-1}\zeta(2s+2s_0-2)\Big\}ds \qquad (7)$$

Proof. This corollary can be obtained by adding the both sides of (6) for $m = k$ and $m = -k$.

§**2.** Let $G = SL_2(\mathbf{Z})$ and G_∞ be the stabilizer of the cusp $i\infty$. The real analytic Eisenstein series $E(z,\alpha)$ is defined by

$$E(z,\alpha) = \sum_{\gamma \in G_\infty \backslash G} (\mathrm{Im}\gamma z)^\alpha \quad for\ \mathrm{Re}\alpha > 1.$$

We put $E^*(z,\alpha) = \xi(2\alpha)E(z,\alpha)$ where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\zeta(s)$ is the Riemann zeta function. It is well known that the function $E^*(z,\alpha)$ has a holomorphic continuation to all α except for simple poles at $\alpha = 0$ and 1 and satisfies the functional equation $E^*(z,\alpha) = E^*(z, 1-\alpha)$. For a natural number k, we put

$$I_k(s;\alpha,f) = \int_0^\infty \int_0^1 E^*(z,\alpha)f(z)y^s e^{2\pi ikx}\frac{dxdy}{y^2},$$

$$\varphi_0(s,\alpha) = \frac{\xi(2\alpha)}{4(\pi k)^{s+\alpha-1/2}}\Gamma(\frac{s+\alpha-1/2+i\kappa}{2})\Gamma(\frac{s+\alpha-1/2-i\kappa}{2})$$
$$+\frac{\xi(2-2\alpha)}{4(\pi k)^{s-\alpha+1/2}}\Gamma(\frac{s-\alpha+1/2+i\kappa}{2})\Gamma(\frac{s-\alpha+1/2-i\kappa}{2}),$$

$$R_k(s;\alpha,f) = \sum_{m=1}^\infty \frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^s}G\Big(\frac{s+i\kappa}{2},\frac{s-i\kappa}{2};s-\alpha+\frac{1}{2};1-(\frac{m+k}{m})^2\Big)$$
$$+\varepsilon_f \sum_{m=1, m\neq k}^\infty \frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^s}G\Big(\frac{s+i\kappa}{2},\frac{s-i\kappa}{2};s-\alpha+\frac{1}{2};1-(\frac{m-k}{m})^2\Big).$$

$G(\alpha,\beta;\gamma;z) = F(\alpha,\beta;\gamma;z) - 1$ as defined in the introduction. The series $R_k(s;\alpha,f)$ converges absolutely for $\mathrm{Re}\, s > 2\mathrm{Re}\,\alpha - 1 + \eta_0$ if $\mathrm{Re}\,\alpha \geq 1/2$, and for $\mathrm{Re}\, s > \eta_0$ if $\mathrm{Re}\, s < 1/2$.

According to [7] Proposition 1, we have

$$D_k(s;\alpha,f) = 4\pi^{s-\alpha+1/2}\Gamma(s-\alpha+\frac{1}{2})\hat{\Gamma}(s-\alpha+1/2; i\kappa, \alpha-1/2)^{-1}$$
$$\times\Big\{I_k(s-\alpha+\frac{1}{2};\alpha,f) - \varepsilon_f\rho(k)\varphi_0(s-\alpha+\frac{1}{2},\alpha)\Big\} - R_k(s;\alpha,f). \quad (8)$$

Theorem 2 . *Let* $\mathrm{Re}(s_0) > 3/2 - \sigma_2$, *then we have*

$$\sum_{n=1}^\infty \rho(n)\Big[Z_{n,k}(s_0) + \varepsilon_f Z_{n,-k}(s_0)\Big]\omega(n)$$

$$= -2i\pi^{s_0-2}\varepsilon_f\Gamma(s_0)\int_{(\sigma_2)}\frac{\pi^{2s}\hat{\omega}(s)I_k(s;s+s_0-1/2,f)}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})\Gamma(s_0+\frac{s-1+i\kappa}{2})\Gamma(s_0+\frac{s-1-i\kappa}{2})\zeta(2s+2s_0-1)}ds$$

$$+2i\pi^{s_0-2}\rho(k)\Gamma(s_0)\int_{(\sigma_2)}\frac{\pi^{2s}\hat{\omega}(s)\varphi_0(s_0;3/2-s-s_0,f)}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})\Gamma(s_0+\frac{s-1+i\kappa}{2})\Gamma(s_0+\frac{s-1-i\kappa}{2})\zeta(2s+2s_0-1)}ds$$

$$-\frac{\varepsilon_f}{2\pi i}\int_{(\sigma_2)}\frac{2^{2s-1}\pi^{2s}\hat{\omega}(s)}{\Gamma(s+i\kappa)\Gamma(s-i\kappa)(\cos\pi s+\cos\pi i\kappa)}\frac{R_k(1-s;3/2-s-s_0,f)}{\zeta(2s+2s_0-1)}ds$$

$$+\frac{\rho(k)}{2\pi i}\int_{(\sigma_2)}\frac{2^{2s-1}\pi^{2s}k^{s-1}\hat{\omega}(s)}{\Gamma(s+i\kappa)\Gamma(s-i\kappa)(\cos\pi s+\cos\pi i\kappa)}\frac{\zeta(2s+2s_0-2)}{\zeta(2s+2s_0-1)}ds$$

Proof. We rewrite the right hand side of Corollary 1 by using (8). Then

$$\sum_{n=1}^{\infty}\rho(n)\big[Z_{n,k}(s_0)+\varepsilon_f Z_{n,-k}(s_0)\big]\omega(n)$$

$$= -2i\pi^{s_0-1}\varepsilon_f\Gamma(s_0)\int_{(\sigma_2)}\hat{\omega}(s)\frac{\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)}{\hat{\Gamma}(s_0;i\kappa,1-s-s_0)}\frac{I_k(s_0;3/2-s-s_0,f)}{\zeta(2s+2s_0-1)}ds$$

$$+2i\pi^{s_0-1}\rho(k)\Gamma(s_0)\int_{(\sigma_2)}\hat{\omega}(s)\frac{\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)}{\hat{\Gamma}(s_0;i\kappa,1-s-s_0)}\frac{\varphi_0(s_0;3/2-s-s_0)}{\zeta(2s+2s_0-1)}ds$$

$$-\frac{\varepsilon_f}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)\frac{R_k(1-s;3/2-s-s_0,f)}{\zeta(2s+2s_0-1)}ds$$

$$+\frac{\rho(k)}{2\pi i}\int_{(\sigma_2)}\hat{\omega}(s)\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)k^{s-1}\frac{\zeta(2s+2s_0-2)}{\zeta(2s+2s_0-1)}ds.$$

Since

$$\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)=\frac{2^{2s-1}\pi^{2s}}{\Gamma(s+i\kappa)\Gamma(s-i\kappa)(\cos\pi s+\cos\pi i\kappa)},$$

$$\frac{\Omega_{i\kappa}(s)(\cos\pi s-\cos\pi i\kappa)}{\hat{\Gamma}(s_0;i\kappa,1-s-s_0)}=\frac{\pi^{2s-1}}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})\Gamma(s_0+\frac{s-1+i\kappa}{2})\Gamma(s_0+\frac{s-1-i\kappa}{2})},$$

and the functional equation $I_k(s;\alpha,f)=I_k(s;1-\alpha,f)$, we get the theorem.

§3. Although we imposed the condition $\mathrm{Re}(s_0)>3/2-\sigma_2$ in the above theorem, we will show that the right hand side can be continued analytically beyond this region. We denote each term in the right hand side of Theorem 2 by J_1,J_2,J_3 and J_4 respectively.

First we consider the term J_2. By the definition of φ_0, the integrand of J_2 is

$$\frac{\pi^{2s-s_0}k^{s-1}\hat{\omega}(s)}{4}\frac{\Gamma(s+s_0-1)\,\Gamma(\frac{1-s+i\kappa}{2})\,\Gamma(\frac{1-s-i\kappa}{2})}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})\Gamma(s_0+\frac{s-1+i\kappa}{2})\Gamma(s_0+\frac{s-1-i\kappa}{2})}$$
$$\times\frac{\zeta(2s+2s_0-2)}{\zeta(2s+2s_0-1)}+\frac{\pi^{3/2-s_0}\hat{\omega}(s)}{4k^{2s_0+s-1}}\frac{\Gamma(s+s_0-1/2)}{\Gamma(\frac{s+i\kappa}{2})\Gamma(\frac{s-i\kappa}{2})}.$$

Let ε be an arbitrary small positive number. We move the line of integration from (σ_2) to $(1-\varepsilon)$. This is possible because there are no poles of the integrand and $\hat{\omega}(s)$ decays rapidly by assumption. Next we will move the point s_0 to the left. J_2 is analytic with respect to s_0 in the region

$$2(1-\varepsilon)+2\mathrm{Re}(s_0)-2>1,$$

that is, $\mathrm{Re}(s_0)>1/2+\varepsilon$. If we want to continue further, we must consider the terms arising from the pole and zeros of the zeta function.

For J_3, we consider in the region $\mathrm{Re}(s+s_0)>1$ to avoid zeros of $\zeta(2s+2s_0-1)$. Then, since $\mathrm{Re}(3/2-s-s_0)<1/2$, $R_k(1-s;3/2-s-s_0,f)$ converges absolutely for $\mathrm{Re}(s)<1-\eta_0$. For arbitrary small $\varepsilon>0$, we put $\sigma_3=1-\eta_0-\varepsilon$ and move the line of integration from (σ_2) to (σ_3). Then we can see that J_3 is holomorphic in $\mathrm{Re}(s_0)>\eta_0+\varepsilon$.

To deal J_1 next,we will define two Poincaré series, namely

$$P_k(z,s)=\sum_{\gamma\in G_\infty\backslash G}(\mathrm{Im}\gamma z)^s e^{2\pi ik\gamma(z)}$$

and

$$\tilde{P}_k(z,s)=\sum_{\gamma\in G_\infty\backslash G}(\mathrm{Im}\gamma z)^s e^{2\pi ik\mathrm{Re}\gamma(z)}.$$

These series are absolutely convergent in $\mathrm{Re}(s)>1$. Furthermore, it is known that $P_k(z,s)$ can be continued to $\mathrm{Re}(s)>1/2$ as a holomorphic function, belongs to $L_2(G\backslash\mathcal{H})$ and $\|P_k(z,s)\|=O(1)$ with fixed $\mathrm{Re}(s)>1/2$. (cf. Hejhal [3], Goldfeld-Sarnak [2].) Since

$$e^{-2\pi ky}=1+(e^{-2\pi ky}-1)=1+O(y)$$

for $y>0$, we have

$$\begin{aligned}P_k(z,s) &= \tilde{P}_k(z,s)+\sum_{\gamma\in G_\infty\backslash G}(\mathrm{Im}\gamma z)^s(e^{-2\pi k\mathrm{Im}\gamma z}-1)e^{2\pi ik\mathrm{Re}\gamma z}\\ &= \tilde{P}_k(z,s)+O\Big(\sum_{\gamma\in G_\infty\backslash G}(\mathrm{Im}\gamma z)^{\mathrm{Re}(s)+1}\Big). \qquad (9)\end{aligned}$$

The last series in (9) converges absolutely for $\mathrm{Re}(s) > 0$ and $O(y^{1+\mathrm{Re}s})$. Now we have

$$\begin{aligned} I_k(s_0; s + s_0 - 1/2, f) &= \xi(2s + 2s_0 - 1) \int_0^\infty \int_0^1 E(z, s + s_0 - 1/2) f(z) y^{s_0} e^{2\pi i k x} \frac{dxdy}{y^2} \\ &= \xi(2s + 2s_0 - 1) \int_{G\backslash\mathcal{H}} E(z, s + s_0 - 1/2) f(z) \tilde{P}_k(z, s_0) \frac{dxdy}{y^2} \end{aligned}$$

So if we move the line of integration (σ_2) of J_1 to (σ) with $\sigma > 1$, we can get the analytic continuation to the region $\mathrm{Re}(s_0) > 1/2$.

We can deal similarly for J_4.

Remarks. 1) We obtained the spectral decomposition of I_k in [7] Lemma 2. Hence,by substituting it for the right hand side of Theorem 2, we can get the spectral decomposition of $\sum_{n=1}^{\infty} \rho(n)\big[Z_{n,k}(s_0) + \varepsilon_f Z_{n,-k}(s_0)\big]\omega(n)$.

2) I have not considered yet how to choose the test function $\omega(x)$. Professor K.Matsumoto pointed out to me that we should consider the test functions which are dependent on s_0,too.

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