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Some Sums involving Farey fractions

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Introduction. For any $x \geqq 1$, let $F_x = F_{[x]}$ denote the sequence of all irreducible fractions with denominators $\leqq x$, arranged in increasing order of magnitude:
$F_x = \left\{ \rho_\nu = \frac{b_\nu}{c_\nu} \middle| 0 < b_\nu \leqq c_\nu \leqq x, (b_\nu, c_\nu) = 1 \right\}$ $\left(\rho_1 = \frac{1}{[x]}, \rho_{\Phi(x)-1} = 1 - \frac{1}{[x]}\right)$, called the Farey series (sequence) of order x.

It is convenient to supplement $\rho_0 = \frac{0}{1} = \frac{b_0}{c_0}$ to F_x to form F'_x because it is then easy to construct F'_{x+1} from F'_x by just inserting all mediants $\frac{b_\nu + b_{\nu+1}}{c_\nu + c_{\nu+1}}$ of successive terms $\frac{b_\nu}{c_\nu}, \frac{b_{\nu+1}}{c_{\nu+1}}$ in F'_x between them as long as $c_\nu + c_{\nu+1} \leqq x+1$. E. g. from $F_2' = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$ we form $F_3' = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$.
The number of terms in the Farey series of order x is

$$\# F_x = \Phi(x) := \sum_{n \leqq x} \varphi(x) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Here, $\varphi(n)$ stands for the Euler function $\sum_{\substack{m \leqq n \\ (m,n)=1}} 1$, the number of integers $\leqq n$ that are relatively prime to n, and is equal to the number of terms in F_x whose denominator $=n$.

By Q_n we denote the set of all pairs of consecutive terms in F'_x:

$$Q_x = \left\{ (c_\nu, c_{\nu+1}) \middle| \frac{b_\nu}{c_\nu} < \frac{b_{\nu+1}}{c_{\nu+1}} \right\}.$$

E. g., $Q_4 = \{(1,3),(3,2),(2,3),(3,1)\}$. $(\# Q_n = \# F_x = \Phi(x))$.

In [7] we considered the sums

$$S_m(x) := \sum_{(c_\nu, c_{\nu+1}) \in Q_x} (c_\nu c_{\nu+1})^{-m} \quad (m \in \mathbf{N})$$

mainly in the special cases m=2, 3. We note the identity

$$\rho_{\nu+1} - \rho_\nu = \frac{b_{\nu+1} c_\nu - b_\nu c_{\nu+1}}{c_{\nu+1} c_\nu} = (c_{\nu+1} c_\nu)^{-1},$$

which follows from the basic relation $b_{\nu+1}c_\nu - b_\nu c_{\nu+1} = 1$. Hence, e.g.

$$S_1(x) = \sum_{\nu=0}^{\Phi(x)-1} (\rho_{\nu+1} - \rho_\nu) = \rho_{\Phi(x)} - \rho_0 = 1 - 0 = 1.$$

In general, $S_m(n)$ can be thought of as the m-th power moment of the differences of consecutive Farey fractions. On these moments the following theorem holds.

Theorem 1. *For $n \to \infty$, we have*

(i) $$S_2(n) = \frac{2}{\zeta(2)n^2}\left\{\log n + \gamma + \frac{1}{2} - \frac{\zeta'}{\zeta}(2)\right\} + \frac{4U(n)\log n}{n^3} + O\left(\frac{\log n}{n^3}\right)$$

(ii) $$S_3(n) = \frac{2\zeta(2)}{\zeta(3)n^3} + \frac{3}{\zeta(2)n^4}\left\{\log n + \gamma - \frac{1}{4} - \frac{\zeta'}{\zeta}(2) + 2\zeta^2(2)c_2^{(1)}(n)\right\} + \frac{12U(n)\log n}{n^5} + O\left(\frac{\log n}{n^5}\right)$$

(iii) $$S_4(n) = \frac{2\zeta(3)}{\zeta(4)n^4} + \frac{1}{n^5}\left\{\frac{4\zeta(2)}{\zeta(3)} + 3\zeta(3)c_3^{(1)}(n)\right\}$$

$$+\frac{20}{3\zeta(2)n^6}\left\{\log n + \gamma - \frac{13}{30} - \frac{\zeta'}{\zeta}(2) + 3\zeta^2(2)c_2^{(1)}(n) + 3\zeta(2)\zeta(3)c_2^{(2)}(n)\right\} + \frac{40U(n)\log n}{n^7} + O\left(\frac{\log n}{n^7}\right)$$

and for $m \geqq 5$

(iv) $$S_m(n) = \frac{2\zeta(m-1)}{\zeta(m)n^m} + \frac{m}{\zeta(m-1)n^{m+1}}\left\{\zeta(m-2) + 2\zeta^2(m-1)c_{m-1}^{(1)}(n)\right\}$$

$$+\frac{m(m+1)}{n^{m+2}}\left\{\frac{\zeta(m-3)}{3\zeta(m-2)} + \zeta(m-2)c_{m-2}^{(1)}(n) + \zeta(m-1)c_{m-2}^{(2)}(n)\right\} + \frac{35\theta'\log n}{\zeta(2)n^{m+3}} + O\left(\frac{1}{n^{m+3}}\right)$$

where γ denotes Euler's constant, θ' is 1 *or* 0 *according as $m = 5$ or $m>5$, and for* $\mathrm{Re}\, s > 1$, $c_s^{(r)}(n)$ *is defined as the absolutely convergent series* $c_s^{(r)}(n) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m}\overline{B}_r\left(\frac{n}{m}\right)$. *Finally,*

$$U(n) := \sum_{m \leqq n} \frac{\mu(m)}{m}\overline{B}_1\left(\frac{n}{m}\right),$$

where $\mu(m)$ denotes the Möbius function, $\overline{B}_1(x)$ denotes the periodic Bernoulli polynomial of degree 1. (*Note that* $\overline{B}_1(0) = -\overline{B}_1(1) = -\frac{1}{2} = B_1$).

Remark 1. Theorem 1 gives very precise description of asymptotic behavior of these sums. Theorem 1 refines fromer results of Mikolas, Hall, Lehner-Newman, Kanemitsu et al and provides a direct relationship between the sums $S_m(n)$ involving Farey fractions and the error term $U(n)$ of the summatory function of Euler's function.

Between Theorem 1 and the results of Maier there is a close connection as follows. We put

$$\delta_\nu = \rho_\nu - \frac{\nu}{\Phi(n)}, \quad \nu = 0,\ldots,\Phi(n)$$

Hence $\delta_{\Phi(n)} = \delta_0 = 0, \delta_1 = \frac{1}{n} - \frac{\nu}{\Phi(n)}$ *etc.*

Also following Maier, we put

$$s_{g,h} = \sum_{\nu=0}^{\Phi(n)-2} (\delta_\nu)^g (\delta_{\nu+1})^h, \quad g, h \geq 0.$$

Noting that $\delta_{\nu+1} - \delta_\nu = (c_\nu c_{\nu+1})^{-1} - \Phi(n)^{-1}, \quad \nu < \Phi(n)$, we have $s_{2,0} - s_{1,1} = \frac{1}{2} S_2(n) - \Phi(n)^{-1}$, whence by Theorem 1, (i) it follows that

$$\sum_{\nu=0}^{\Phi(n)-1} (\delta_\nu)^2 = s_{2,0}(= s_{0,2}) = s_{1,1} + \frac{1}{\zeta(2)n^2}\left(\log n + \gamma + \frac{1}{2} - \frac{\zeta'}{\zeta}(2)\right) - \frac{1}{2\zeta(2)n^2} + O\left(\frac{\log n}{n^3}\right)$$

$$= s_{1,1} + \frac{1}{\zeta(2)n^2}\left(\log n + \gamma + \frac{1}{2} - \frac{\zeta'}{\zeta}(2)\right) + O\left(\frac{\log n}{n^3}\right).$$

The estimate of $s_{2,0}$ should be difficult. In fact, it was known to Franel that

$$s_{2,0} = \sum_{\nu=0}^{\Phi(n)-1} (\delta_\nu)^2 = O(n^{-1+\varepsilon}) \Leftrightarrow RH.$$

However, regarding $s_{2,1}$, Theorem 1 already gives a very precise asymtotic formula:

$$s_{2,1} = \frac{1}{6} S_3(n) + O(n^{-4}) = \frac{\zeta(2)}{3\zeta(3)n^3} + O(n^{-4}).$$

Corollary. *The identity*

$$\sum_{j=0}^{m-2} \sum_{r=2}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{2\binom{j+m}{j+1}}{r^{m+j+1} k^{m-j-1}} = 1$$

holds. Combining this with the reciprocity laws of Sitaramachandrarao and Sivaramasarma [14], [15], *we obtain identities like*

(1) $$\sum_{r=1}^{\infty} r^{-3} \sum_{k=1}^{r} k^{-1} = \frac{5}{4}\zeta(4),$$

(2) $$\sum_{r=1}^{\infty} r^{-4} \sum_{k=1}^{r} k^{-2} = \frac{945\zeta^2(3)}{\pi^6} - \frac{1}{3}.$$

Remark 2. (1) represents the value at $s = 3$ of the zeta-function $H(s) = \sum_{n=1}^{\infty} n^{-s} \sum_{k=1}^{n} k^{-1}$ considered by Matsuoka [12], while (2) represents the value at $s = 4, z = 2$ of the zeta-function $H(s,z) = \sum_{n=1}^{\infty} n^{-s} \sum_{k=1}^{n} k^{-z}$ considered by Apostol-Vu [1]. Zagier [16] has proved that the values of multiple zeta-functions can in general be expressed as linear combinations of products of values of the Riemann zeta-function and this completely settles these types of problems save for concrete determination of the coefficients.

Theorem 2 (Refinment of a theorem of Mikolás [13]). *We have for* $n \to \infty$,

(i) $$\sum_{\nu=1}^{\Phi(n)} \rho_\nu^{-a} = \frac{\zeta(a)}{(a+1)\zeta(a+1)} n^{a+1} - \frac{1}{a-1}\Phi(n) - \zeta(a)c_a^{(1)}(n)n^a + O_a\left(n^{\max\{a-1,1\}}\log^\theta n\right),$$

where $\theta = \theta(a) = 0,\ a \neq 2,\ \ \theta(2) = 1$.

(ii) $$\sum_{\nu=1}^{\Phi(n)} \rho_\nu^{-1} = \Phi(n)\left\{\log n + \gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\}^a + O(n\log n).$$

Theorem 3 (Refinment of a theorem of Hand-Dumir [2]). *We have for* $n \to \infty$,

$$\sum_{(k,k')\in Q_n} \frac{1}{k^2 k'} = \frac{1}{\zeta(2)n}\left\{\log n + \gamma + 1 - \frac{\zeta'(2)}{\zeta(2)}\right\} + \frac{4U(n)\log n}{n^2} + O\left(\frac{\log n}{n^2}\right).$$

Theorem 3 is a special case of the formula for $\sum_{(k,k')\in Q_n} \frac{1}{k^a k'}$ $(a \in \mathbf{N})$.

Theorem 4 (Refinment of a theorem of Kanemitsu [7], which in turn is a refinement of that of Lehner-Newman [10]).

(i) *For* $0 \leqq a, b$,

$$\sum_{(k,k')\in Q_n} k^a (k')^b = c_{a,b} n^{a+b+2} - \frac{\pi^2}{6}(a+b+2)c_{a,b}U(n)n^{a+b+2} + O\left(n^{a+b+1}(\log n)^{\theta''}\right),$$

where $\theta'' = 1$, *if* $0 < b \leqq \frac{1}{2}$ *and* $\theta'' = 0$ *if* $b = 0, b > \frac{1}{2}$, *and*

$$c_{a,b} = \frac{6}{\pi^2}\left\{\frac{1}{(1+a)(1+b)} - \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(3=a+b)}\right\}.$$

(ii) $$\sum_{(k,k')\in Q_n} \frac{1}{kk'(k+k')} = \frac{12\log 2}{\pi^2 n} + \frac{2\log 2}{n^2}U(n) + O\left(\frac{1}{n^2}\right);$$

$$\sum_{(k,k')\in Q_n} \frac{1}{k+k'} = \frac{6}{\pi^2}(2\log 2 - 1)n + (1 - 2\log 2)U(n) + O(1);$$

$$\sum_{(k,k')\in Q_n} \frac{kk'}{k+k'} = \frac{11 - 12\log 2}{3\pi^2}n^3 + (11 - 12\log 2)\frac{nE(n)}{6} + O(n^2),$$

where $E(n) = \Phi(n) - \frac{3}{\pi^2}n^2$.

Remark 3. In the course of proof of Theorem 4, (ii), we reprove Gupta's identity

$$\sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^2(r+k)} = \frac{3}{4}.$$

This can be obatined also as the case $a = 1$ of Formula (19) for the zeta-function

$$T^*(s,z) := \sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^s k^z (r+k)} = \frac{1}{\zeta(s+z+1)} T(s,z)$$

considered by Apostol-Vu [1] (with $T(s,z)$ the zeta-function defined by Formula (16)).

Hata [6] developed a generalization of Farey fractions to deduce general summation formulas including (1), (2) and Gupta's identity. Also, his theorem enables one to express Euler's constant in terms of an infinite series that contains only rational numbers. For the purpose of finding the value of infinite series, Hata's method [6] is simpler and has wider range of applications.

§ 2. Supplementary lemmas

In this section we collect lemmas which together with those lemmas in [7] can provide proofs of the theorems stated in §1.

Lemma 1'. For any $u \in \mathbf{C}$ and any $x \to \infty$, let $L_u(x) := \sum_{n \le x} n^u$. Then for any $\ell \in \mathbf{N}$ with $\ell > \mathrm{Re}\, u + 1$ we have

$$L_u(x) = \begin{cases} \frac{1}{u+1} x^{u+1} + \zeta(-u), & u \neq -1 \\ \log x + \gamma, & u = -1 \end{cases} + \sum_{r=1}^{\ell} \frac{(-1)^r}{r} \binom{u}{r-1} B_r(x) x^{u+1-r} + O\left(x^{\mathrm{Re}\, u - \ell}\right).$$

In the special case where $u \in \mathbf{N} \cup \{0\}$, we can take $\ell = u+1$ without error term, in which case it is convenient to write the formula in the form

$$L_u(x) = \frac{1}{u+1} \sum_{r=0}^{u+1} (-1)^r \binom{u+1}{r} \overline{B}_r(x) x^{u+1-r} + \zeta(-u).$$

For instance, for k=0, Lemma 1' just says $L_0(x) = x - \overline{B}_1(x) - \frac{1}{2}$.

In the course of proof of Lemma 1, we prove the identity for any $\ell \in \mathbf{N}$ with $\ell > -\mathrm{Re}\, s$

(3) $$\zeta(s) = A_{-s}(\ell) = 1 + (-1)^{\ell+1} \binom{-s}{\ell} \int_1^\infty \overline{B}_\ell(t) t^{-s-\ell} dt - \frac{1}{1-s} \sum_{r=0}^{\ell} (-1)^r \binom{1-s}{r} B_r$$
$$= \frac{1}{s-1} + \frac{1}{2} + \frac{1}{12} s + \cdots + (-1)^{\ell+1} \binom{-s}{\ell} \int_1^\infty \overline{B}_\ell(t) t^{-s-\ell} dt$$

The Cauchy criterion shows that the the integral on the RHS represents an analytic function for $\mathrm{Re}\, s > -\ell$, so that for $\mathrm{Re}\, s > -\ell$, $\zeta(s)$ is analytic with the excception of a simple pole at $s = 1$. Since $\ell \in \mathbf{N}$ is arbitrary, this signifies that $\zeta(s)$ is extended analytically over the whole plane with the excception of a simple pole at $s = 1$. We can, however, proceed still further to give a **conceptually the simplest proof** of the functional equation of the Riemann zeta-fucntion. Actually, although in a well-known proof one considers the case $\ell = 1$ of Formula (3) to use the Lebesgue dominated convergence theorem on the grounds that the partial sums of the Fourier series of $\overline{B}_1(t)$ are bounded, we can prove the following corollary by just employing absolutely convergent Fourier series in the case $\ell = 2$ of Formula (3) without resorting such a rather high-brow theorem.

Corollary. *The Riemann zeta-function satisfies the functional equation*

$$\boxed{\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi}{2}s\zeta(1-s).}$$

The defining Dirichlet series for $\zeta(1-s)$ being absolutely convergent for $\mathrm{Re}\, s < 0$, this gives an analytic continuation of $\zeta(s)$ into $\mathrm{Re}\, s < 0$ in explicit form.

Proof. In the case $l = 2$ the above formula has the form

$$\zeta(s) = 1 - \binom{-s}{2}\int_1^\infty \overline{B}_2(t)t^{-s-2}dt + \frac{1}{s-1} + \frac{1}{2} + \frac{1}{12}s$$

The improper integral $\int_0^1 \overline{B}_2(t)t^{-s-2}dt$ can be calculated when $-2 < \mathrm{Re}\, s < -1$, to give

$$\int_0^1 \overline{B}_2(t)t^{-s-2}dt = \frac{1}{1-s} + \frac{1}{s} - \frac{1}{6}\frac{1}{s+1}.$$

Hence in the same region $-2 < \mathrm{Re}\, s < -1$, we get the representation

$$\zeta(s) = -\frac{s(s+1)}{2}\int_0^\infty \overline{B}_2(t)t^{-s-2}dt.$$

Substituting the **absolutely convergent** Fourier series $\frac{1}{\pi^2}\sum_{n=1}^\infty \frac{\cos 2\pi nt}{n^2}$ for $\overline{B}_2(t)$, we get

$$\zeta(s) = -\frac{s(s+1)}{2}\frac{1}{\pi^2}\sum_{n=1}^\infty \int_0^\infty t^{-s-2}\cos 2\pi nt dt$$

By the change of variables $2\pi nt = u$ this becomes further

$$= -2(2\pi)^{s-1}s(s+1)\int_0^\infty u^{-s-2}\cos u du \sum_{n=1}^\infty n^{1-s}.$$

Hence by a formula for Mellin transform

$$\int_0^\infty u^{-s}\cos u du = \sin\frac{\pi}{2}s\Gamma(1-s)$$

we conclude that

$$\zeta(s) = -2(2\pi)^{s-1}s(s+1)\left(-\sin\frac{\pi}{2}s\right)\Gamma(-1-s)\zeta(1-s)$$

$$= -2(2\pi)^{s-1}s(-s-1)\Gamma(-1-s)\sin\frac{\pi}{2}s\zeta(1-s) = -2(2\pi)^{s-1}(-s)\Gamma(-s)\sin\frac{\pi}{2}s\zeta(1-s),$$

whence we get the formula in question. □

We continue to state further lemmas. As immediate corollaries of Lemma 1', we have

Lemma 11. *For any $a \in \mathbf{R}$ and any $r \in \mathbf{N}$ we have*

$$L_a^*(x,r) := \sum_{\substack{n \le x \\ (n,r)=1}} n^a = \begin{cases} \frac{\phi(r)}{r}\frac{x^{a+1}}{a+1} + \zeta(-a)\sum_{d|r}\mu(d)d^a, & a \ne -1 \\ \frac{\phi(r)}{r}(\log x + \alpha(r) + \gamma), & a = -1 \end{cases}$$

$$+\sum_{k=1}^{\ell}\frac{(-1)^k}{k}\binom{a}{k-1}x^{a+1-k}\sum_{d|r}\mu(d)\overline{B}_k\left(\frac{x}{d}\right)d^{k-1}+O\left(x^{a-\ell}\sigma_\ell(r)\right),$$

where $\alpha(r)$ is defined before.

Proof $L_a^*(x,r)=\sum_{n\le x}\sum_{d|(n,r)}\mu(d)=\sum_{d|r}\mu(d)d^aL_a\left(\frac{x}{d}\right)=\sum_{d|r}\mu(d)d^aL_a\left(\frac{x}{d}\right)$

$$=\sum_{d|r}\mu(d)d^a\begin{cases}\frac{1}{a+1}\left(\frac{x}{d}\right)^{a+1}+\zeta(-a), & a\neq -1\\ \log\frac{x}{d}+\gamma, & a=-1\end{cases}+\sum_{k=1}^{\ell}\frac{(-1)^k}{k}\binom{a}{k-1}\overline{B}_k\left(\frac{x}{d}\right)\left(\frac{x}{d}\right)^{a+k-1}$$

$$+O\left(\left(\frac{x}{d}\right)^{a-\ell}\right)$$

$$=\begin{cases}\frac{x^{a+1}}{a+1}\sum_{d|r}\frac{\mu(d)}{d}+\zeta(-a)\sum_{d|r}\mu(d)d^a, & a\neq -1\\ (\log x+\gamma)\sum_{d|r}\frac{\mu(d)}{d}-\sum_{d|r}\frac{\mu(d)\log d}{d}, & a=-1\end{cases}+\sum_{k=1}^{\ell}\frac{(-1)^k}{k}\binom{a}{k-1}x^{a+1-k}\sum_{d|r}\mu(d)\overline{B}_k\left(\frac{x}{d}\right)d^{k-1}$$

$$+O\left(x^{a-\ell}\sum_{d|r}d^\ell\right),$$

whence the result follows. □

Corollary. *For any $a\in\mathbf{R}$, with $\mathrm{N}\ni\ell>a+1$*

$$\sum_{\nu=1}^{\Phi(x)}\rho_\nu^{\,a}=\begin{cases}\frac{1}{a+1}\Phi(x)+\zeta(-a)S_{-a}(x), & a\neq -1\\ \sum_{n\le x}\varphi(n)(\log n+\alpha(n))+\gamma\Phi(x), & a=-1\end{cases}+\sum_{\substack{r=1\\2|r}}^{\ell}\frac{(-1)^r}{r}\binom{a}{r-1}B_rS_{a-1}(x)+O(x).$$

Proof. Substituting for $L_a^*(n)=L_a^*(n,n)$ from lemma 11 in $\sum_{\nu=1}^{\Phi(x)}\rho_\nu^{\,a}=\sum_{n\le x}\frac{1}{n^a}L_a^*(n)$, we get

$$\sum_{\nu=1}^{\Phi(x)}\rho_\nu^{\,a}=\sum_{n\le x}\frac{1}{n^a}\begin{cases}\frac{\varphi(n)}{n}\frac{n^{a+1}}{a+1}+\zeta(-a)\sum_{d|n}\mu(d)d^a, & a\neq -1\\ \frac{\varphi(n)}{n}(\log n+\alpha(n)+\gamma), & a=-1\end{cases}$$

$$+\sum_{n\le x}\frac{1}{n^a}\sum_{r=1}^{\ell}\frac{(-1)^r}{r}\binom{a}{r-1}n^{a+1-r}\sum_{d|n}\mu(d)\overline{B}_r\left(\frac{n}{d}\right)d^{a-1}+O\left(\sum_{n\le x}n^a\sigma_{a-\ell}(n)\right)$$

$$=\begin{cases}\frac{1}{a+1}\sum_{n\le x}\varphi(n)+\zeta(-a)\sum_{n\le x}\sum_{d|n}\mu(d)\left(\frac{d}{n}\right)^a, & a\neq -1\\ \sum_{n\le x}\varphi(n)(\log n+\alpha(n)+\gamma), & a=-1\end{cases}+\sum_{r=1}^{\ell}\frac{(-1)^r}{r}\binom{a}{r-1}B_r\sum_{n\le x}\sum_{d|n}\mu(d)\left(\frac{n}{d}\right)^{a-1}$$

$$+O\left(\sum_{n\le x}\sigma_{-\ell}(n)\right)$$

$$=\begin{cases}\dfrac{1}{a+1}\Phi(x)+\zeta(-a)\displaystyle\sum_{n\le x}d^a\mathrm{M}\left(\frac{x}{n}\right), & a\ne -1\\ \displaystyle\sum_{n\le x}\varphi(n)(\log n+\alpha(n))+\gamma\Phi(x), & a=-1\end{cases}+\sum_{\substack{r=1\\2|r}}^{\ell}\frac{(-1)^r}{r}\binom{a}{r-1}B_r\sum_{d\le x}d^{1-a}\mathrm{M}\left(\frac{x}{d}\right)+O(x),$$

whence the result follows. □

Lemma 12. *For the summatory function of the arithmetical function* $t(k):=\sum_{d|k}\frac{\mu(d)}{d}\log\frac{k}{d}$ *we have as* $x\to\infty$

$$\sum_{k\le x}t(k)=\frac{x}{\zeta(2)}\left\{\log x-1-\frac{\zeta'(2)}{\zeta(2)}\right\}+H(x)\log x+O(\log x).$$

where

$$H(x)=\sum_{k\le x}\phi(k)-\frac{x}{\zeta(2)}.$$

This lemma is an improvement of Lemma 8 in [7].

Lemma 13. *For* $a>1$,

$$\sum_{r=n+1}^{\infty}\frac{1}{r^a}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{k}=\frac{1}{(a-1)\zeta(2)n^{a-1}}\left\{\log n+\gamma+\frac{1}{a-1}-\frac{\zeta'(2)}{\zeta(2)}\right\}+\frac{U(n)}{n^a}\log n+O\left(\frac{\log n}{n^a}\right).$$

Lemma 14. *For* $a\ge 2, b\ge 2$,

$$\sum_{r=n+1}^{\infty}\frac{1}{r^a}\sum_{d|r}\frac{\mu(d)}{d^b}=\frac{1}{(a-1)\zeta(b+1)n^{a-1}}+\frac{c_b^{(1)}(n)}{n^a}+\frac{(1-\alpha)ac_{b-1}^{(2)}(n)}{2n^{a+1}}+O\left(\frac{(\log n)^{\beta}}{n^{a+2-\alpha}}\right),$$

where $\alpha=\left[\frac{2}{b}\right]$, $\beta=\left[\frac{3}{b}\right]$.

Lemma 15. *For* $a\ge 2, b\ge 2$,

$$\sum_{r=n+1}^{\infty}\frac{1}{r^a}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{k^b}=\frac{\zeta(b)}{(a-1)\zeta(b+1)n^{a-1}}+\frac{1}{n^a}\left\{\frac{c_b^{(1)}(n)}{n^a}-\frac{\alpha}{a\zeta(2)}\right\}+\frac{(1-\alpha)}{2n^{a+1}}\left\{a\zeta(b)c_{b-1}^{(2)}(n)-\frac{(b-2)\beta}{(\alpha+1)\zeta(2)}\right\}$$

$$+O\left(\frac{(\log n)^{\beta}}{n^{a+2-\alpha}}\right).$$

Lemma 16. *Putting* $A_j(n):=2\binom{j+m}{j+1}\sum_{r=2}^{m}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}}$, *we can express* $S_m(n)$ *as*

$$S_m(n)=1-\sum_{j=0}^{m-2}A_j(n).$$

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