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Eisenstein cocycles for arithmetic groups and values of zeta functions

Robert Sczech (九州大理)

Let F be a totally real number field of degree n over $\mathbb{Q}$, and f a conductor of a ray class group in F. By definition, $f=f_{\infty}f_{\text{fin}}$ is the product of the finite part f_{fin} which is an integral ideal of $\mathbb{Z}_F$, and the infinite part $f_{\infty}=\prod\mathfrak{P}_i$, where $\mathfrak{P}_i$ runs through a set of embeddings of F into $\mathbb{R}$, indexed by a subset $S\subseteq\{1,2,\cdots,n\}$. Let $I(f)$ be the multiplicative group of fractional ideals in F generated by all prime ideals in $\mathbb{Z}_F$ which do not divide f_{fin}. Two ideals $a,b\in I(f)$ belong to the same class mod f iff ab^{-1} is a principal ideal (α) generated by an element $\alpha\in 1+f_{\text{fin}}b^{-1}$ such that $\mathfrak{P}_i(\alpha)>0$ for all $i\in S$. Modulo this relation, $I(f)$ decomposes into finitely many classes C mod f. To every class C there is associated the partial zeta function

$$\zeta(C,s)=\sum_{a\in C}N(a)^{-s},\ \ \mathrm{Re}(s)>1.$$

According to Hecke, this function has an analytic continuation to the whole complex s-plane except for a simple pole at s=1, and, by results of Klingen and Siegel, the special values of $\zeta(C,s)$ at non-positive integral $s=0,-1,-2,\ldots$ are all rational numbers which can be calculated explicitely using a well known formula of Shintani. In the simplest case, this is the classical formula of Euler,

$$\zeta(1-k)=-\frac{B_k}{k}\ ,\ k=1,2,3,\ldots,$$

for the special values of the Riemann zeta function $\zeta(s)$. Since the Bernoulli numbers B_k of an odd index $k>1$ are all zero, it follows that $\zeta(-2k)=0$ for k=1,2,3,.... This is in fact a general phenomenon. Because of Gamma factors in Hecke's functional equation, $\zeta(C,s)$ vanishes at $s=-2k$ of order

$$\operatorname*{ord}_{s=-2k}\ \zeta(C,s)\geq r=n-|S|\ ,\ k=0,1,2,\ldots.$$

In particular, $\zeta(C,-2k)=0$ if $r>0$. It is therefore of interest to investigate the coefficients

$$\zeta^{(r)}(C,-2k)=\frac{d^r}{ds^r}\ \zeta(C,s)\Big|_{s=-2k}.$$

For instance, these numbers are the subject of the well known conjectures of Stark (k=0) and Beilinson-Gross ($k>0$). In this report, we are interested in the cohomogical interpretation of these values in terms of the group cohomology of the unit group

$$U=\{\eta\in\mathbb{Z}_F\mid\eta\in 1+f_{\text{fin}},\ \mathfrak{P}_i(\eta)>0\text{ for all }i\in S\}.$$

It is convenient to assume that U is torsionfree. Then, according to Dirichlet, U is a free abelian group of rank $n-1$, and therefore, the homology as well as the cohomology groups of U are isomorphic to the (co)homology of the torus T^{n-1}, $T=\mathbb{R}/\mathbb{Z}$. In particular, the homology group $H_{n-1}(U,\mathbb{Z})$ is free abelian of rank one, so we can talk about a fundamental class Z of U, which is a generator of $H_{n-1}(U,\mathbb{Z})$. (In the case n=2, Z corresponds to a fundamental unit of U).

Theorem 1. There is a cohomology class $\varepsilon_p(C,k) \in H^{n-1}(U,\mathbb{R})$ such that the evaluation on Z gives

$$\zeta^{(p)}(C,-k) = \varepsilon_p(C,k)(Z)$$

provided that either $p=0$ and k=1,3,5,... or $p=n-|S|$ and k=0,2,4.... Moreover, ε_p is the restriction of a universal Eisenstein cohomology class in $H^{n-1}(GL_n\mathbb{Z})$ which depends only on n and p, but not on the particular field F or ray class C.

This is a generalization of a previous result [1] which deals with the special case p=0. In that case, it can be shown that the cohomology class $\varepsilon_0(C,k)$ is in fact rational, $\varepsilon_0(C,k) \in H^{n-1}(U,\mathbb{Q})$. Moreover, a finite formula exists for $\varepsilon_0(C,k)$ which generalizes the classical Dedekind sum. In general, our method does not lead to any conclusion about the arithmetic nature of the cohomology classes $\varepsilon_p(C,k)$ for $p>0$. The proof of the above theorem will be published elsewhere. In this report, we wish to illustrate the construction of the Eisenstein cocycle in the simplest non-trivial case: n=2, p=1, k even.

Let $G=GL_2\mathbb{R}$ and H be the the subspace of homogenous polynomials in $\mathbb{R}[x_1,x_2]$. The set $M=\{f: H\times\mathbb{R}^2 \to \mathbb{C}\}$ is then a G-module under the action

$$(Af)(P,x) = \det(A)\, f(A^tP, xA) \quad , A\in G \quad , \quad f\in M.$$

Here, A^tP denotes the polynomial defined by $(A^tP)(y)=P(yA^t)$. We first construct a homogenous 1−cocycle ψ for G with values in M. By definition, ψ is a map $\psi: G\times G\to M$ satisfying the properties

$$\psi(A_1,A_2)+\psi(A_2,A_3)=\psi(A_1,A_3), \qquad (1)$$

$$\psi(AA_1,AA_2) = A\psi(A_1,A_2)\ ;\ A, A_j\in G. \qquad (2)$$

For $A_i\in G$, we denote the jth column of the matrix A_i by A_{ij}. Then the cocycle ψ is defined for $x\neq 0$ by

$$\psi(A_1,A_2)(P,x) = P(\partial_{x_1},\partial_{x_2})\Big(\frac{\det(A_{11},A_{21})}{<x,A_{11}><x,A_{21}>}\Big) \qquad (3)$$

where $P(\partial_{x_1},\partial_{x_2})$ denotes the differential operator formed with the partial derivatives with respect to x_1 and x_2. The definition needs a modification if one of the scalar products $<x,y> = x_1y_1+x_2y_2$ in the denominator vanishes. For instance, if $<x,A_{11}>=0$, then $<x,A_{12}>\neq 0$ since $x\neq 0$; assuming that the second scalar product $<x,A_{21}>$ in (3) does not vanish, the right side of (3) must be replaced in that case by

$$P(\partial_{x_1},\partial_{x_2})\Big(\frac{\det(A_{12},A_{21})}{<x,A_{12}><x,A_{21}>}\Big).$$

A similar modification applies in all other cases except when x=0 in which case we set ψ=0. For details of this construction and the proof that the so defined map ψ does indeed represent a cohomology class in $H^1(G,M)$, we refer the reader to [1].

The basic idea behind the construction of the Eisenstein cocycle $\varepsilon=\varepsilon_1$ is to average the values of ψ with respect to the variable x over the lattice $\mathbb{Z}^2$. Let $\Gamma=GL_2\mathbb{Z}$ and let N be the set of complex valued functions $f(P,Q,u,v)$ on $H\times H\times\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})^2$. N is a left Γ-module under the action

$$(Af)(P,Q,u,v) = \det(A) f(A^t P, A^{-1}Q, A^{-1}u, A^{-1}v).$$

For $A_i \in \Gamma$, the Eisenstein cocycle ε is the map $\varepsilon : \Gamma \times \Gamma \rightarrow N$ defined by ($\mathbf{e}(z) = \exp(2\pi i z)$)

$$\varepsilon(A_1, A_2)(P,Q,u,v) \underset{\text{def}}{=} \sum_{x \in \mathbb{Z}^2} \operatorname{sign}(xu)\, \mathbf{e}(-xv)\, \psi(A_1,A_2)(P,x) \Big|_Q ,$$

where the "Q-limit" notation on the right has to be understood as

$$\sum_x h(x) \Big|_Q \underset{\text{def}}{=} \lim_{t \to \infty} \Big(\sum_{|Q(x)|<t} h(x) \Big).$$

Theorem 2. The map $\varepsilon : \Gamma \times \Gamma \rightarrow N$ is well defined and has the properties

$$\varepsilon(A_1,A_2) + \varepsilon(A_2,A_3) = \varepsilon(A_1,A_3) \ , \ A_i \in \Gamma$$
$$\varepsilon(AA_1,AA_2) = A\varepsilon(A_1,A_2) \ , \ A \in \Gamma.$$

Moreover, ε represents a non trivial cohomology class in $H^1(\Gamma, N)$.

For the proof, see [2]. We return now to the partial zeta function of the introduction and consider the case of a real quadratic field F with one distinguished real embedding $\mathfrak{P}: F \rightarrow \mathbb{R}$ such that $f_\infty = \mathfrak{P}$. Let $b \in C$ be a fixed representative of the ray class C and choose a $\mathbb{Z}$-basis W for $f_{\text{fin}} b^{-1} = \mathbb{Z}W_1 + \mathbb{Z}W_2$. The trace form in F determines the dual basis V by $\operatorname{tr}(V_i W_j) = \delta_{ij}$. Define P, Q, v by

$$P(x) = \mathrm{N}(\sum x_i W_i) \ , \ Q(x) = \mathrm{N}(\sum x_i V_i) \ , \ v_j = \operatorname{tr}(V_j), \ j{=}1,2.$$

P and Q are normforms determined by the bases W resp. V. Finally, let $A \in \Gamma$ be the hyperbolic matrix corresponding to a generator of U under the regular representation of U with respect to the basis V. Then, as a special case of of Theorem 1, we have the explicit relation

$$\zeta'(C,-2k) = \pm (2\pi i)^{-1-4k} \varepsilon(1,A)(P^{2k}, Q, \mathfrak{P}(V), v).$$

The sign ambiguity is due to the fact that the right side changes its sign when A is replaced by A^{-1}.

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References

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Kyushu University 33
Fukuoka 812, Japan

Rutgers University
Newark NJ 07102, USA

email: sczech@math.kyushu-u.ac.jp