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A remark on almost uniform distribution modulo 1

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Let (a_n), $n = 1, 2, \ldots$ be a sequence of real numbers and $A(I, (a_n), N)$ be the *counting function*, that is, the number of $n = 1, 2, \ldots, N$ that $\{a_n\}$ is contained in a certain interval $I \subset [0, 1]$. Here we denote by $\{a_n\} = a_n - [a_n]$, the fractional part of a_n. First we recall a kind of generalization of the classical definition of uniform distribution modulo 1 (see [11], [3] and [10]).

Definition. The sequence (a_n) is said to be *almost uniformly distributed modulo* 1 (abbreviated a.u.d. mod 1) if there exist a strictly increasing sequence of natural numbers (n_j), $j = 1, 2, \ldots$ and, for every pair of a, b with $0 \le a < b \le 1$,

$$\lim_{j \to \infty} \frac{A([a, b), (a_n), n_j)}{n_j} = b - a.$$

The purpose of this note is to emphasize the usefulness of this concept in considering the oscillation problems in number theory.

Now, for example, we define (c_n) by

$$c_n = \frac{n}{2^{1+[\log_2 n]}}.$$

Then (c_n) is a.u.d. mod 1 but not u.d. mod 1. It is obvious that if the sequence (a_n) is u.d. mod 1, then a.u.d. mod 1. On the contrary, if

$$n_{j+1} - n_j = o(n),$$

then a.u.d. mod 1 implies u.d. mod 1. According to the classical method of uniform distribution theory (see e.g. [8]), we can show the following

Proposition 1. The sequence (a_n), $n = 1, 2, \ldots$ is a.u.d. mod 1 if and only if there exist a strictly increasing sequence of natural numbers (n_j), $j = 1, 2, \ldots$, and for every real-valued continuous function on the interval $[0, 1]$, we have

$$\lim_{j \to \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} f(\{a_i\}) = \int_0^1 f(x) dx.$$

Proposition 2. **(Weyl's Criterion for a.u.d. mod** 1**)** The sequence (a_n), $n = 1, 2, \ldots$ is a.u.d. mod 1 if and only if there exist a strictly increasing sequence of natural numbers (n_j), $j = 1, 2, \ldots$, and for every integer h, we have

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \exp(2\pi h\sqrt{-1}\{a_i\}) = 0.$$

We should pay attention to the next generalization of Fejér's Theorem.

Theorem 1. **(Fejér's Theorem for a.u.d. mod** 1**)** Let $(f(n))$, $n = 1, 2, \ldots$ be a sequence of real numbers and $\Delta f(n) = f(n+1) - f(n)$. If the following three conditions is satisfied, then $(f(n))$ is a.u.d. mod 1:

1. There exists a natural number N that $\Delta f(n)$ is monotone when $n \geq N$ (hereafter, we say this property as *ultimately monotone*),
2. $\lim_{n\to\infty} \Delta f(n) = 0$,
3. $\limsup_{n\to\infty} n|\Delta f(n)| = \infty$.

Note that the corresponding third conditions for u.d. mod 1 is:

$$\lim_{n\to\infty} n|\Delta f(n)| = \infty.$$

Moreover, it is shown in [7] that $\limsup_{n\to\infty} n|\Delta f(n)| = \infty$ is the necessity condition for u.d. mod 1 (see also [6]). Concerning this fact, in [3], it is shown that $(\log n)$ is not a.u.d. mod 1 but a.u.d. mod 1 in the "average" sense. It is an interesting problem to study this delicate difference between u.d. mod 1 and a.u.d. mod 1. We can show the following:

Corollary 1. Let $(g(n))$ be a sequence of real numbers which satisfies three conditions:

(C1) $g(n) = o(n)$,

(C2) Let $f(n) = \frac{1}{n}\sum_{k=1}^{n} g(k)$, then $f(n)$ is *not* a.u.d. mod 1,

(C3) $\limsup_{n\to\infty} |f(n) - g(n+1)| = \infty$.

Then $\Delta^2 f(n)$ changes its sign infinitely many times. Here $\Delta^2 f(n) = \Delta(\Delta f(n))$.

Proof. We have

$$\begin{aligned}\Delta f(n) &= \frac{1}{n+1}\sum_{k=1}^{n+1} g(k) - \frac{1}{n}\sum_{k=1}^{n} g(k) \\ &= \frac{1}{n+1} g(n+1) - \frac{1}{n(n+1)}\sum_{k=1}^{n} g(k). \end{aligned} \tag{1}$$

This shows that $\lim_{n\to\infty} \Delta f(n) = 0$. And by (1),

$$(n+1)\Delta f(n) = g(n+1) - f(n).$$

Thus

$$\limsup_{n\to\infty} n|\Delta f(n)| = \infty.$$

If $\Delta f(n)$ is ultimately monotone, then $f(n)$ is a.u.d. mod 1, which contradicts with the assumption. □

Let P_n be the n-th prime and we will later apply this Corollary 1 for the oscillation problem of P_n. Now we show

Theorem 2. $(\log P_n)$ is not a.u.d. mod 1.

Proof. Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the real torus, which is identified with the interval $[0, 1)$ via the map $x \to \{x\}$. Define by χ_τ, the characteristic function of $[\tau, \tau + 1/2)$ mod $\mathbf{Z}$ in $\mathbf{T}$, and by $\pi(x)$, the number of primes smaller or equal to x. Let us evaluate, for a positive integer k,

$$T_\tau(k) = \frac{1}{\pi(N_k)} \sum_{n=1}^{\pi(N_k)} \chi_\tau(\log P_n),$$

with $N_k = e^{k+\tau+1/2}$. By using the prime number theorem of the form:

$$\pi(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x}),$$

we have, for a positive number $c = o(k)$,

$$\begin{aligned}\pi(N_k)T_\tau(k) &= \sum_{n=1}^{\pi(N_{k-c})} + \sum_{n=\pi(N_{k-c+1})}^{\pi(N_k)} \\ &= \frac{e^{k+\tau+1/2}}{k+\tau+1/2} - \frac{e^{k+\tau}}{k+\tau} + \frac{e^{k+\tau-1/2}}{k+\tau-1/2} - \frac{e^{k+\tau-1}}{k+\tau-1} + \dots \\ &\quad + \frac{e^{k+\tau+3/2-c}}{k+\tau+3/2-c} - \frac{e^{k+\tau+1-c}}{k+\tau+1-c} + O(\frac{e^{k-c}}{k-c}) + O(\frac{c\cdot e^k}{(k-c)^2}).\end{aligned}$$

If $c = [2\log k]$, then

$$\begin{aligned} T_\tau(k) &= 1 - \frac{k+\tau+1/2}{e^{1/2}(k+\tau)} + \frac{k+\tau+1/2}{e(k+\tau-1/2)} - \frac{k+\tau+1/2}{e^{3/2}(k+\tau-1)} + \dots \\ &\quad + \frac{k+\tau+1/2}{e^{c-1}(k+\tau+3/2-c)} - \frac{k+\tau+1/2}{e^{c-1/2}(k+\tau+1-c)} + O(\frac{\log k}{k}) \\ &= 1 - e^{-1/2} + e^{-1} - e^{-3/2} + \dots + e^{-c+1} - e^{-c+1/2} + O(\frac{\log k}{k}) \\ &= \frac{\sqrt{e}}{1+\sqrt{e}} + O(\frac{\log k}{k}). \end{aligned}$$

Here, the implied constant of the last O symbol does not depend on the choice of $\tau \in [0,1)$, and we have $\sqrt{e}/(1+\sqrt{e}) = 0.622459\ldots > 1/2 + \delta$ with a positive constant δ. This shows that there exist an open neighborhood of τ in $\mathbf{T}$:

$$U_\tau = \{x \in \mathbf{T} | d(x,\tau) < \epsilon\}$$

that if $\{\log N\} - 1/2 \in U_\tau$, then

$$\frac{1}{N}\sum_{n=1}^{N} \chi_\tau(\log P_n) > 1/2 + \delta, \tag{2}$$

for sufficiently large N. Here $d(x,y)$ is the natural distance between x and y on $\mathbf{T}$. Remark that the value ϵ does not depend on the choice of k, if k is sufficiently large. As $\mathbf{T}$ is compact, there exist a finite sub covering $\mathbf{T} \subset \cup_{i=1}^{m} U_{\tau_i}$. Thus there exist a constant M that if $N \geq M$, then we have (2). If the sequence $(\log P_n)$ is a.u.d. mod 1, then by Proposition 1, there exist a strictly increasing sequence of natural numbers (n_j), $j = 1, 2, \ldots$ that

$$\lim_{j\to\infty} \frac{1}{n_j}\sum_{i=1}^{n_j} \chi_\tau(\{\log P_i\}) = \frac{1}{2},$$

which is a contradiction. □

Now we give a very different proof of the results of [2].

Theorem 3. $\Delta^2 \log P_n$ changes its sign infinitely many times.

Proof. Let $g(n) = n\log P_n - (n-1)\log P_{n-1}$ and $f(n) = \log P_n$ in Corollary 1. (Here we put $P_0 = 1$ for example.) By using Theorem 2, it suffice to show (C1) and (C3). By using prime number theorem, we have

$$\log P_n = 1 + \frac{P_n}{n} + O(\frac{1}{\log P_n}). \tag{3}$$

Thus we see

$$\begin{aligned} g(n) &= P_n - P_{n-1} + 1 + O(\frac{n}{\log n}) \\ &= o(n). \end{aligned}$$

Here we used the fact:

$$P_n - P_{n-1} = O(P_n^{\theta}),$$

with a certain positive constant $\theta < 1$. This type of result was first shown by G. Hoheisel in [4] with $\theta = 1 - 1/33000 + \epsilon$. The best knowledge up to now is $\theta = 23/42$ in [5]. For the condition (C3),

$$\begin{aligned} g(n+1) - f(n) &= (n+1)(\log P_{n+1} - \log P_n) \\ &> \frac{(n+1)(P_{n+1} - P_n)}{P_{n+1}} \\ &\sim \frac{P_{n+1} - P_n}{\log P_n}. \end{aligned}$$

Here we write $f \sim g$ if $|f/g| \to 1$. P. Erdös [1] was the first to obtain

$$\limsup_{n\to\infty} \frac{P_{n+1} - P_n}{\log P_n} = \infty,$$

by showing

$$\limsup_{n\to\infty} \frac{(P_{n+1} - P_n)(\log\log\log P_n)^2}{\log P_n \log\log P_n \log\log\log\log P_n} > \exists c > 0.$$

About the improvement of the constant c, see [9]. This completes the proof. □

Our method to show this type of results can be generalized by a kind of "linearity" in many cases. To explain this, we notice

Theorem 4. Let l be a fixed positive integer, and C_i $(i = 1, 2, \ldots, l)$ be the real numbers with $\sum C_i \neq 0$. The sequence $(\sum_{i=0}^{l-1} C_i \log P_{n+i})$ is not a.u.d. mod 1.

Proof. First, we consider the case $(C \log P_n)$. Without loss of generality, we may assume that $C > 0$. Then we write $C \log P_n = \log_b P_n$ with a constant $b > 1$. To see the assertion, replace e with b in the proof of Theorem 2.

If $l > 1$, it suffice to note that

$$\sum_{i=0}^{l-1} C_i \log P_{n+i} - \log P_n \sum_{i=0}^{l-1} C_i = o(1).$$

This shows the assertion. □

Theorem 5. Let l be a fixed positive integer, and f_i $(i = 1, 2, \dots, l)$ be the positive real numbers. Then

$$\Delta^2 \log(P_n^{f_1} P_{n+1}^{f_2} \dots P_{n+l-1}^{f_l})$$

changes its sign infinitely many times.

Proof. Put

$$\begin{aligned} g(n) &= n(\sum_{i=1}^{l} f_i \log P_{n+i-1}) - (n-1)(\sum_{i=1}^{l} f_i \log P_{n+i-2}) \\ f(n) &= \sum_{i=1}^{l} f_i \log P_{n+i-1}. \end{aligned}$$

By using Corollary 1 and Theorem 4, in a similar manner as in the proof of Theorem 3, we see the assertion. Here, we essentially used the positiveness of f_i $(i = 1, 2, \dots l)$ in proving (C3). □

We expect that the conditions $f_i > 0$ $(i = 1, 2, \dots l)$ can be droped.

Our method is applicable to a lot of arithmetic functions $g(n)$, that $f(n) = 1/n \sum_{k \le n} g(k)$ is not a.u.d. mod 1. For example, we can show similar assertions for the divisor function $d(n) = \sum_{d|n} 1$ as

$$\frac{1}{n} \sum_{k=1}^{n} d(k) = \log n + (2\gamma - 1) + O(\frac{1}{\sqrt{n}}),$$

with the Euler constant γ. The proof for this case is easier, but the results do not seem well worthy of stating here.

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