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ON THE SOLUTIONS OF THUE EQUATIONS

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Let $k/\mathbb{Q}$ be a finite extension, $p(X,Y) \in k[X,Y]$ a homogeneous polynomial of degree $n > 3$ with non-zero discriminant, and $a \in k^{\times} = k \setminus \{0\}$. The equation

$$p(X,Y) = aZ^n$$

defines a regular curve C in $\mathbb{P}_k^2$ of genus $g = (n-1)(n-2)/2$.

Here we obtain an estimate for the number of integral solutions and a certain information about rational points.

Remark 0.1. In [2], we required that $p(X,Y)$ be divisible by a linear element in $k[X,Y]$, but it is easily seen we do not have to assume that. The same results follow if we replace the map f^a there by the map $f\colon C \to J =$ the Jacobian of C,

$$C(\bar{k}) \ni P \mapsto \mathcal{O}_C((2g-2)P - \text{a canonical divisor}) \in \operatorname{Pic}^0(C_{\bar{k}}) \simeq J(\bar{k})$$

which is defined over k, noting that this map equals $2g-2$ times f^a.

1. Integral solutions

For simplicity, we state the result only in the case of rational integral solutions. As for the algebraic S-integer version, we refer the reader to [2, Theorem 5.4].

In this section we assume that a and the coefficients of $p(X,Y)$ are in $\mathbb{Z}$.

In 1983, Silverman used the Jacobian variety J of C to estimate the number of integral solutions from above:

Theorem 1.1 (Silverman [4]**).** *If all the exponents of prime factors of a are less than n, and $|a|$ is sufficiently large, then*

$$\#\{(x,y) \in \mathbb{Z}^2 \mid p(x,y) = a\} < n^{2n^2}(8n^3)^R,$$

where $R = \operatorname{rank} J(\mathbb{Q})$.

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We can think of $J(\mathbb{Q}) \otimes_{\mathbf{Z}} \mathbb{R}$ as a Euclidean space with some height function. He mapped $C(\mathbb{Q})$ to $J(\mathbb{Q})$ and counted the number of lattice points which lie in a ball of $J(\mathbb{Q}) \otimes_{\mathbf{Z}} \mathbb{R}$.

On the other hand, Mumford had asserted in 1965 paper [3] that in general, the heights of rational points on the Jacobian which come from a curve under a certain map increase at least exponentially if the genus is greater than 1.

Putting the above two results together, we obtain

Theorem 1.2. *If all the exponents of prime factors of a are less than n, and $|a|$ is sufficiently large, then*

$$\#\{(x,y) \in \mathbb{Z}^2 \mid p(x,y) = a\} \leq 4 \cdot 7^R,$$

where $R = \operatorname{rank} J(\mathbb{Q})$.

2. Rational points

For a general curve C over k of genus $g > 1$, there is a result of Vojta about the distribution of the rational points of C in the Jacobian variety J.

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be respectively the inner product and the norm on $J(k) \otimes_{\mathbf{Z}} \mathbb{R}$ induced by the Néron-Tate height attached to the Θ-divisor.

Theorem 2.1 (Vojta [5], cf. [1]). *Assume $C(k) \neq \emptyset$. Regard $C \subset J$ by an appropriate map. For $\varepsilon > 0$, there exists a constant $\gamma = \gamma(C, \varepsilon)$ such that if $P, Q \in C(k)$ satisfy the inequalities $\|P\| > \gamma$ and $\|Q\| > \gamma \|P\|$, then*

$$\langle P, Q \rangle / \|P\| \|Q\| < 1/\sqrt{g} + \varepsilon.$$

If we have a non-trivial automorphism of the curve, what can we say about the distribution of the rational points of the curve? When asking this question, we prefer to use the morphism $f \colon C \to J$ given by

$$C(\bar{k}) \ni P \mapsto \mathcal{O}_C((2g-2)P - \text{a canonical divisor}) \in \operatorname{Pic}^0(C_{\bar{k}}) \simeq J(\bar{k}),$$

where $\bar{k}$ is an algebraic closure of k and $C_{\bar{k}} = C \times_k \operatorname{Spec} \bar{k}$. Automorphisms of C induce norm preserving morphisms of J compatible with the above map. In other words, there exists a canonically defined representation of $\operatorname{Aut}_k C$ on the Euclidean space $(J(k) \otimes_{\mathbf{Z}} \mathbb{R}, \|\cdot\|)$ which leaves the image of $C(k)$ stable.

In the case of Thue curves, we obtain the following result: Let C be the Thue curve as before. For an n-th root ζ of unity in k, we have an automorphism of C defined as

$$C(\bar{k}) \ni P = (x : y : z) \mapsto P_\zeta := (x : y : \zeta z) \in C(\bar{k}).$$

Proposition 2.2. *For $P \in C(k)$, if $\zeta \neq 1$ and $\|fP\| \neq 0$, then*

$$\langle fP_\zeta, fP\rangle / \|fP_\zeta\| \|fP\| = -1/(n-1).$$

As an application of this kind of results, if C is a twisted Fermat curve of degree 4, for example, we can see the rational points lie in an intersection of quadric hypersurfaces in $J(k) \otimes_{\mathbf{Z}} \mathbb{R}$. The author will explain it elsewhere.

References

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