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A Topological Approach to the Complex Hénon Family

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Abstract

In this talk I explain some topological descriptions for the Julia sets of some polynomial automorphisms of $\mathbb{C}^2$ including complex Hénon mappings, which are recently developed by Bedford and Smillie. They showed that, if the mapping is hyperbolic and its Lyapunov exponent is minimal, then the "Julia set for backward iteration" $J_- \setminus K_+$ is modeled by a simple Riemann surface lamination and the Julia set itself is expressed as a quotient space of the solenoid. A relationship between the Lyapunov exponent and "dynamical critical points" is also mentioned.

1 Motivations

Let us consider a *generalized Hénon mapping* on $\mathbb{C}^2$ with the following form:

$$f_{p,b} \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} p(x) - by \\ x \end{pmatrix} \quad b \in \mathbb{C}^*,$$

where $p(x)$ is a monic polynomial of one variable with complex coefficients and degree not smaller than two. Complicated but interesting dynamical properties of this mapping (with all parameters being real and $p(x)$ being a quadratic polynomial, which will be simply called a *Hénon mapping*) restricted to $\mathbb{R}^2$ were first observed numerically by Hénon [H] for a special choice of parameters.

Some of the motivations to study the Hénon mappings may be the followings:

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Fatou-Bieberbach Domains: Example of a *Fatou-Bieberbach domain* in $\mathbb{C}^2$ which is a domain biholomorphically equivalent to $\mathbb{C}^2$ whose complement has non–empty interior [B, F2] is given by an Hénon mapping (consider an Hénon mapping with at least two attractive cycles; each attractive basin is equivalent to $\mathbb{C}^2$). Notice that this phenomena can never be found in one–dimension due to the Liouville's theorem. The boundary of this domain seems to have a very complicated structure, in fact it often fails to be a topological manifold.

Strange Attractors: For suitable choices of real parameters, this mapping on $\mathbb{R}^2$ has a *strange attractor* (i.e. an attractor with very complicated shape and sensitive dynamics on it) as Hénon observed [H]. This kind of observation is mathematically justified (but in different situations from that of Hénon) by Benedicks and Carleson [BC], Mora and Viana [MV], and recently by Jakobson and Newhouse [JN].

Classification: Friedland and Milnor [FM] (and Hénon [H] for degree two case) proved that a polynomial automorphism f on $\mathbb{C}^2$ or $\mathbb{R}^2$ (i.e. a polynomial map with a constant Jacobian determinant which has again a polynomial map as its inverse) with non–trivial dynamics is linearly conjugated to a mapping which is a composition of some generalized Hénon mappings:

$$f = f_{p_1,b_1} \circ f_{p_2,b_2} \circ \cdots \circ f_{p_k,b_k}.$$

In the following, we consider a mapping f of the form as above with degree $d \geq 2$ so that the dynamics is interesting. Here, by the *degree* of f, one means the product of the degree d_i of p_i which will be denoted by d. Remark that the Jacobian determinant of f is $b \equiv b_1 \cdots b_k$.

2 Cast of Players

According to Hubbard and Oberste-forth [HO1], we focus on the following sets:

$$U_\pm \equiv \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \left\| f^{\pm n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow +\infty \ (n \to +\infty) \right\},$$

$$K_\pm \equiv \mathbb{C}^2 \setminus U_\pm, \quad \text{and} \quad J_\pm \equiv \partial K_\pm.$$

Futher, we define

$$J \equiv J_+ \cap J_-,$$

and call it the *Julia set* of f. One can easily see that each set is invariant and J is compact. To investigate these sets the following *Green functions* are fundamental:

$$G_\pm \begin{pmatrix} x \\ y \end{pmatrix} \equiv \lim_{n \to \infty} \frac{1}{d^n} \log^+ \left\| f^{\pm n} \begin{pmatrix} x \\ y \end{pmatrix} \right\|.$$

It is shown that $G_\pm$ vanish on $K_\pm$, positive and pluri-harmonic on $U_\pm$, and have dynamical compatibility:

$$G_\pm \circ f\begin{pmatrix} x \\ y \end{pmatrix} = d^{\pm 1} \cdot G_\pm\begin{pmatrix} x \\ y \end{pmatrix}.$$

The following construction is very useful for our purpose:

$$\varphi_+\begin{pmatrix} x \\ y \end{pmatrix} \equiv \lim_{n\to\infty}\left(q_1 \circ f^n\begin{pmatrix} x \\ y \end{pmatrix}\right)^{\frac{1}{d^n}},$$

where q_1 means the projection to the first coordinate and the branch of d^n-th root is taken so that φ_+ tangents to q_1 when $\|(x,y)\|$ goes to infinity. One then proves that this is well-defined on

$$V_+ \equiv \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |x| \geq |y|,\ |x| \geq R\right\},$$

for sufficiently large $R > 0$ and analytic there. This function also has dynamical compatibility:

$$\varphi_+ \circ f\begin{pmatrix} x \\ y \end{pmatrix} = \left(\varphi_+\begin{pmatrix} x \\ y \end{pmatrix}\right)^d.$$

In the works of Bedford and Smillie [BS1, BS2, BS3] (also with Lyubich [BLS1, BLS2], Fornaess and Sibony [FS]), they considered $(1,1)$-currents defined by

$$\mu_\pm \equiv dd^c G_\pm.$$

Moreover, it is proven that their "product":

$$\mu \equiv \mu_+ \wedge \mu_-$$

is a well-defined $(2,2)$-current and becomes a measure. Using these stuffs, they showed many important results corresponding to the Julia–Fatou–Brolin theory including some topological properties of $J_\pm$ and $K_\pm$, density of saddle periodic points in the support of μ (this corresponds to the fact that, in one–dimensional case, the repelling periodic cycles are dense in the Julia set: in two–dimension, the support of μ coincides with the Julia set when the dynamics is hyperbolic), and some characterizations of μ in terms of the thermodynamical formalism (more precisely, μ is shown to be the unique maximal entropy measure: thus, μ is a natural generalization of the one–dimensional Brolin measure).[1]

[1]Further analogie is true: the one–dimensional Brolin measure is characterized as a limit distribution of the backward images of a generic starting point. A similar "convergence theorem to the maximal entropy measure starting from a simple current" was established in two–dimension (but for diffeomorphisms!), and this is one of the main tools in the theories of Bedford(–Lyubich)–Smillie, Fornæss–Sibony etc. For more informations on these potential theoritic methods, see Nishimura [N]

3 A Topological Approach

The methods presented in the paper [BS4] are more topological. To clarify the meaning of their results, let us briefly review the corresponding facts in one–dimension.

One–dimensional Theory (see, for example, [Mi])

Let p be a monic polynomial on $\mathbb{C}$. By definition, the *Julia set* J_p of p is the boundary of the *filled–in Julia set*:

$$K_p \equiv \left\{ z \in \mathbb{C} \;\middle|\; |p^n(z)| \text{ does not tend to infinity when } \; n \to +\infty \right\}.$$

The first important result is a relationship between the behavior of the critical points of p and the connectivity of its Julia set.

Theorem 3.1 (Fatou [F1], Julia[J]) *The Juila set of a polynomial p is connected if and only if all critical points of p remain bounded by the iteration.*

When one extends the polynomial to the Riemann sphere by putting $p(\infty) = \infty$, the dynamics of p near infinity almost look like

$$p_0 : z \longmapsto z^d.$$

In fact, one can find a holomorphic bijection φ from a neighborhood of infinity to a neiborhood of infinity which conjugates p_0 to p (called the Bötkher coordinate [Mi]). This conjugacy is unique if it is assumed to tangent to the identity at infinity. Moreover, one can extend this map as a holomorphic bijection from the compliment of the closed unit disc $\overline{\Delta}$ to the compliment of K_p if and only if all critical points of p are in K_p. One of the strongest tools in combinatorial studies of the Julia set is the following *external ray of angle θ*:

$$\varphi(\{re^{2\pi i\theta} | r > 1\}).$$

It is shown that, if p is hyperbolic, then all external rays can be extended to $r \geq 1$. So finally we have a surjection:

$$\varphi : \partial\Delta = S^1 \longrightarrow J_p,$$

which conjugates the dynamics p_0 to p. Thus, we can topologically produce the Julia set as a quotient space of S^1:

$$J_p = S^1/_\sim,$$

where we set $\theta_1 \sim \theta_2$ if and only if $\varphi(\theta_1) = \varphi(\theta_2)$, and the action of p on J_p is the induced map of p_0 on the quotient space. In the degree 2 case, this equivalence relation is well understood combinatorialy.

Now it seems quite natural to ask the following questions for Hénon mappings:

- What is the best topological model for a connected Julia set ? What should we prepare instead of S^1 ?
- By what does the Bötkher coordinate should be replaced ? How to define the external rays ?
- A priori a Hénon mapping does not have a critical point because it is a diffeomorphism. Still, can one define "dynamical critical points" (in some sense) for a Hénon mapping ?

In fact, the answer to the first question in hyperbolic case given by Bedford and Smillie is the *solenoid* (see also the next section):

$$\begin{aligned}\Sigma &\equiv \varprojlim(S^1,\ z \longmapsto p_0(z) = z^d) \\ &\equiv \Big\{(\cdots, z_{-2}, z_{-1}, z_0) \,\Big|\, z_n \in S^1,\ p_0(z_{i-1}) = z_i\ (i \leq 0)\Big\}.\end{aligned}$$

Solenoid first appeared in this context in [HO1].

Theorem 3.2 (Hubbard–Oberste-forth [HO1]) *For a generalized Hénon mapping H there exists a compactification X of $\mathbb{C}^2$ adding S^3 at infinity such that*

(1) the induced topologies on $\mathbb{C}^2$ and S^3 are the standard ones,

(2) $\mathbb{C}^2$ is dense in X,

(3) the closure of J_- in X is $J_- \sqcup \Sigma$,

(4) the mapping H can be continuously extended to S^3 so that its restriction to Σ is $\hat{p}_0$ given by:

$$\hat{p}_0(\cdots, z_{-2}, z_{-1}, z_0) \equiv (\cdots, z_{-3}, z_{-2}, z_{-1}).$$

Recall that, in one–dimensional case, if one makes a compactification of $\mathbb{C}$ by adding S^1 instead of a point at infinity, then any polynomial extends continuously to S^1 and the restriction on it is the multiplication of the angles by the degree of the polynomial. This is why we used the unit circle and the angle multiplication on it to make a topological model of the one–dimensional Julia set. So one may expect that there exists a homeomorphism:

$$\Psi\ : \Sigma_+ \equiv \varprojlim(\mathbb{C} \setminus \overline{\Delta},\ z \longmapsto p_0(z)) \longrightarrow\ J_- \setminus K_+,$$

which conjugates the dynamics $\hat{p}_0$ on the projective limit to f on $J_- \setminus K_+$ where, again, $\hat{p}_0$ means the standard lift of p_0 to Σ_+ and Ψ is hopefully holomorphic on each leaf. This is realized the first thing for the cases of small perturbations of hyperbolic one–dimensional polynomials in [HO2].

4 Results of Bedford and Smillie [BS4]

One of the theorems of Bedford and Smillie states that the above observation is true if the Julia set is connected and the dynamics on it is hyperbolic. Moreover, they gave answers to othe questions mentioned above. To state their results, let us recall the definition of the *Lyapunov exponent*:

$$\Lambda_\mu(f) \equiv \lim_{n\to\infty} \frac{1}{n} \int \log \|D^n f(x)\| d\mu(x).$$

This means the rate of maximal expansion of f avaraged by μ. By definition, f is said to be *solenoidal* if its Lyapunov exponent is minimal (this means that $\Lambda_\mu(f) = \log d$. See Theorem 4.4 below).

The concept of the solenoidal mapping is tightly connected to the existence of *dynamical critical points*:

$$C^u \equiv \bigcup_{p\in J} C^u_p,$$

where C^u_p means the critical points of G_+ in U_+ restricted to an open subset of the unstable manifold of p.

Some of their resluts in [BS4] are summarized as:

Theorem 4.1 (Bedford–Smillie [BS4]) *Let f be hyperbolic on J. Then the following conditions are equivalent.*

(i) f is solenoidal.

(ii) C^u is empty.

(iii) There exists a homeomorphism:

$$\Psi \; : \; \Sigma_+ \longrightarrow J_- \setminus K_+$$

which conjugates the dynamics and it is holomorphic on each leaf.

(iv) J is connected.

Corollary 4.2 (Bedford–Smillie [BS4]) *In the situation as above, Julia set is expressed as a quotient space of the solenoid, i.e., there exists a surjective continuous map:*

$$\Psi : \Sigma \longrightarrow J$$

which conjugates the dynamics $\hat{p}_0$ to f. Moreover, this map is bounded to one.

Sketch of the Proof of Theorem 4.1 We start with one more analogy between the dynamics in dimension one and two. Let us recall the definition of the *Lyapunov exponent* for a one dimensional polynomial p:

$$\lambda_\nu \equiv \lim_{n\to\infty} \frac{1}{n} \int \log |D^n p(x)| d\nu(x),$$

where ν is the harmonic measure of J_p.

Theorem 4.3 (Manning [Ma], Przytycki [P]) *For an one–dimensional monic polynomial p of degree d, one has*

$$\lambda_\nu = \log d + \sum G(c_i)$$

where c_i are the critical points of p and G is the Green function of J_p.

Bedford and Smillie obtained a similar formula for a polynomial automorphism f. To state their result, let us define a measure which expresses the distribution of the critical points. For any *Pesin box* $B = B^s \cap B^u$ (i.e., $B^s = \bigcup_{e\in E} \Gamma^s_e$ where Γ^s_e is a transversal to a holomorphic bidisk such that $\Gamma^s_e \subset W^s(x)$ for some $x \in J$. B^u is defined in a similar fashion), the restriction μ on B is a product measure $\mu^u \otimes \mu^s$ with respect to the topological structure of B, where μ^u (resp. μ^s) means the transversal measure in the unstable (resp. stable) direction. We define the *critical measure* as

$$\mu_c^-(X) \equiv \int_{e\in E} \sharp\{\mathcal{C}_e \cap X\} d\mu^s(e),$$

where $\mathcal{C}_e$ is the set of critical points of G^+ on $\Gamma^s_e \setminus K_+$.

Theorem 4.4 (Bedford–Smillie [BS4]) *For every $t > 0$, we have*

$$\Lambda_\mu = \log d + \int_{t\le G^+<td} G^+ d\mu_c^-.$$

Thus, f is solenoidal if and only if $\Lambda_\mu(f) = \log d$.

¿From the theorem above, we can deduce that $\Lambda_\mu(f) = \log d$ if and only if G^+ has no critical point on $W^u(p) \setminus K_+$ for μ-almost all $p \in J$. Now, let us assume that G^+ has no critical point on $W^u(p) \setminus K_+$ and let $\mathcal{O}$ be a connected component of $W^u(p) \setminus K_+$. Then, it is not difficult to see that $\mathcal{O}$ is simply connected and unbounded (because G^+ is pluriharmonic on U_+). If a point in $\mathcal{O}$ is sufficiently far from p, then it is an element of V_+. Thus, we can make an analytic continuation of φ_+ to $J_- \setminus K_+$. Let us define

$$\Phi \ : \ J_- \setminus K_+ \longrightarrow \Sigma_+,$$

by

$$\Phi(x,y) \equiv (\cdots, \varphi_+(f^{-2}(x,y)), \varphi_+(f^{-1}(x,y)), \varphi_+(x,y)).$$

It is shown that, by the hyperbolicity, this map is a covering of finite degree, say m. Define $\pi \ : \ \Sigma_+ \longrightarrow \mathbb{C} \setminus \overline{\varDelta}$ by $\pi(\cdots, z_{-2}, z_{-1}, z_0) = z_0$.

For a complex number $s \in \mathbb{C}$ we write $s = \Re s + i\Im s$ where $\Re s$ and $\Im s$ denote the real and imaginary parts of s. Let us define a holomorphic vector field $S \equiv s(\partial_{\Re s} - i\partial_{\Im s})$. On $\mathbb{C}\setminus\overline{\varDelta}$ we consider two vector fields $\Re S$ and $\Re iS$, and let X and Y be their lifts to $J_- \setminus K_+$ by $\Phi \circ \pi$. Integral curves of X are called *external rays.*

Let $S_0 \subset \Sigma_+$ consist of points with 0-th coordinate equal to the real number e. Put $T_0 \equiv (\Phi)^{-1}(e) \subset J_- \setminus K_+$ and let

$$\Phi_0 \,:\, T_0 \longrightarrow S_0$$

be the restriction of Φ to T_0. It is shown that, if a partition $\mathcal{P}$ of T_0 is sufficiently fine, then $(\Phi_0|_P)^{-1}$ is bijective on each classes $P \in \mathcal{P}$. So, if a partition $\mathcal{Q}$ of S_0 is sufficiently fine, then $\Psi_0 \equiv \Phi_0^{-1} \circ \Theta_m$ is a bijection on each classes $Q \in \mathcal{Q}$, where

$$\Theta_m \,:\, \Sigma \longrightarrow \Sigma$$

is defined by

$$\Theta_m(\cdots, z_{-2}, z_{-1}, z_0) \equiv (\cdots, z_{-2}^m, z_{-1}^m, z_0^m).$$

We extend this mapping Ψ_0 using $\mathbb{R}$-actions induced by Y and $\Re iS$, and next $\mathbb{R}$-actions induced by X and $\Re S$. Modifying this map a little bit, we finally get the global conjugacy map Ψ.

If f is hyperbolic on J, then all external rays land on J. And the extended map:

$$\Psi \,:\, \Sigma \longrightarrow J$$

is a continuous surjection. Thus, J is connected.

In [BLS1] they showed that connectivity of J implies the minimality of the Lyapunov exponent. This completes the proof of the theorem.

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