Parabolic fixed points of two dimensional complex dynamical systems

(2 次元複素力学系の放物型不動点について)

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0. Introduction

Let $T$ be a holomorphic mapping of a neighborhood, $V$, of the origin, $O = (0, 0) \in \mathbb{C}^2$, into $\mathbb{C}^2$ with $T(O) = O$. The germ of such a mapping is called a local analytic transformation.

Let $\mathcal{T}$ denote the set of all local analytic transformations. Local analytic transformations $T$ and $T'$ are said to be $r$-equivalent if their power series expansion at the origin coincide up to order $r$. The equivalence class is called the $r$-jet of the local analytic transformation.

Local analytic transformations $T$ and $T'$ are said to be $r$-conjugate if there is an invertible local analytic transformation $S$ such that $S^{-1} \circ T \circ S$ and $T'$ are $r$-equivalent. Let $\mathcal{T}_I = \{T \in \mathcal{T} | dT(O) = id\}$, where $dT$ denotes the differential of $T$ and $id$ denotes the identity map. The elements of $\mathcal{T}_I$ are called parabolic local analytic transformations. Ueda[2] gave a classification of 2-jets of $\mathcal{T}_I$.

Let $E = \{P \in \mathbb{C}^2 \mid T^n(P) \to O \text{ as } n \to \infty\}$, and $D = \{P \in \mathbb{C}^2 \mid T^n \text{ converge uniformly to } O \text{ in some neighborhood of } P \text{ as } n \to \infty\}$. If $D \neq \emptyset$, then we say $O$ has a basin of attraction.

In Ueda’s list of normal forms, the case of $N_{2,1}(\lambda)$ ( case I-B in our classification ) :

\begin{align*}
(0.1) \quad \begin{cases} 
  x_1 &= x + \lambda x^2 + xy + \cdots \\
  y_1 &= y + (\lambda + 1)xy + y^2 + \cdots 
\end{cases}
\end{align*}

has a parabolic basin if $\text{Re}(\lambda) > 0$. In this note, we shall prove that the fixed point of the above type has another attractive basin of a different
type. The author does not know if they are analytically conjugate or not in the basins. Since this new type of attractive basin appears as a degenerate case of parabolic basin, we call such a basin a weakly-parabolic basin.

1. 2-jets of parabolic local analytic transformations

Let $f: \mathbb{C}^2 \to \mathbb{C}$ and $g: \mathbb{C}^2 \to \mathbb{C}$ be homogeneous polynomials of degree 2, and let $F: \mathbb{C}^2 \to \mathbb{C}^2$ be a parabolic analytic transformation defined by

$$F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + f(x, y) \\ y + g(x, y) \end{array} \right).$$

Let $H: \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$H \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} f(x, y) \\ g(x, y) \end{array} \right)$$

denote the homogeneous part of degree 2. We have $F = id + H$.

If an invertible local analytic transformation $S$ has a linear part $L \in GL(2, \mathbb{C})$, then the 2-jet of $S^{-1} \circ F \circ S$ is given by

$$L^{-1} \circ F \circ L = id + L^{-1} \circ H \circ L.$$ 

Hence, if parabolic local transformations $F = id + H$ and $F' = id + H'$ are 2-equivalent, then there exists a linear isomorphism $L \in GL(2, \mathbb{C})$ such that

$$L^{-1} \circ H \circ L = H'$$

and vice versa. Thus, the classification of 2-jets is reduced to the classification of homogeneous polynomial maps $H: \mathbb{C}^2 \to \mathbb{C}^2$ under the conjugacy $L^{-1} \circ H \circ L$ with $L \in GL(2, \mathbb{C})$. We have several cases.

- **CASE I**: $f(x, y)$ and $g(x, y)$ are mutually prime.
- **CASE II**: $f(x, y)$ and $g(x, y)$ have a common factor of degree one.
- **CASE III**: $f(x, y)$ or $g(x, y)$ is a scalar multiple of the other (and not both zero).
- **CASE IV**: both $f(x, y)$ and $g(x, y)$ are 0.

First, let us consider the case I. Let $\pi: \mathbb{C}^2 \setminus \{O\} \to \overline{\mathbb{C}}$ denote the natural projection of $\mathbb{C}^2 \setminus \{O\}$ to the Riemann sphere $\overline{\mathbb{C}}$. Homogeneous maps $H$ and $H'$ induce rational maps of degree 2 on the Riemann
sphere. We denote the induced rational maps by $[H] : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $[H'] : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ respectively.

**Lemma 1.1** $H$ and $H'$ are conjugate by an element of $GL(2, \mathbb{C})$ if and only if $[H]$ and $[H']$ are conjugate by a Möbius transformation.

The classification of rational functions of degree 2 under the conjugacy of Möbius transformations is well known (e.g. see Milnor[1]). A conjugacy class of rational functions of degree two is characterized by the set of three multipliers of the fixed points. The three multipliers, say $\mu_1, \mu_2, \mu_3$, are subject to the restriction

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0.$$  

These values are invariant under the conjugacies.

If $\mu_i \neq 1 (i = 1, 2, 3)$, then the residues at each of the fixed points

$$\lambda_i = \frac{1}{2\pi\sqrt{-1}} \int \frac{dz}{[H](z) - z} = \frac{1}{\mu_i - 1}$$

give another set of holomorphic invariants. The values $\lambda_i$ are called "translation numbers" in the normal forms studied by Ueda[2]. $\lambda_1, \lambda_2, \lambda_3$ are subject to the restriction

$$\lambda_1 + \lambda_2 + \lambda_3 = -1.$$  

Ueda[2] proved the following.

**Theorem (Ueda)** If $\text{Re} \lambda_i > 0$, then $F$ has a (parabolic) basin of attraction of the fixed point $O$ which corresponds to $\lambda_i$. If $F$ is an automorphism of a complex manifold, then the basin of attraction is isomorphic to $\mathbb{C}^2$ and the dynamics in the basin is analytically conjugate to a translation.

This theorem holds also in the cases I-B and II-A-2 below. See Ueda[2] for the proof. Our case I is divided into three sub-cases.

**Case I-A :** $[H]$ has three distinct fixed points.

**Case I-B :** $[H]$ has a double fixed point and a simple fixed point.

**Case I-C :** $[H]$ has a triple fixed point.

Normal forms as 2-jets for these cases are as follows.

(I-A) \[
\begin{align*}
  x_1 & = x + \lambda_1 x^2 + (\lambda_2 + 1)xy \\
  y_1 & = y + (\lambda_1 + 1)xy + \lambda_2 y^2.
\end{align*}
\]
Note that in our case I-A, we exclude the case where $\lambda_i = 0$ holds for some $i$. This case is treated as case II-A-1 and III-A-1, since in this case the components of $H$ have a common factor.

The parameter $\lambda$ in the following normal form is given by $\lambda = \frac{1}{\mu_1-1}$, if $\mu_1 \neq 1$ and $\mu_2 = \mu_3 = 1$, for example.

(I-B) \[
\begin{align*}
x_1 &= x + \lambda x^2 + xy \\
y_1 &= y + (\lambda + 1)xy + y^2.
\end{align*}
\]

Note that in our case I-B, we exclude the case of $\lambda = 0$, in which case the induced map $[H]$ degenerates to a Möbius transformation with an indeterminate point. This case will be treated as case II-B-1.

In case I-C, we have $\mu_1 = \mu_2 = \mu_3 = 1$.

(I-C) \[
\begin{align*}
x_1 &= x + xy \\
y_1 &= y + x^2 + y^2.
\end{align*}
\]

Next, consider the case II, where $f(x,y)$ and $g(x,y)$ have a common factor and the induced map $[H]$ defines a Möbius transformation except at the indeterminate point corresponding to the common factor. We have three possibilities for the Möbius transformation $[H]$.

Case II-A : $[H]$ has two distinct fixed points.
Case II-B : $[H]$ has a double fixed point.
Case II-C : $[H]$ is the identity.

And taking the indeterminate point, originating from the common factor, into considerations, we have sub-cases as follows.

Case II-A-1 : the indeterminate point is different from the fixed points.
Case II-A-2 : the indeterminate point coincides with one of the fixed points of the Möbius transformation.
Case II-B-1 : the indeterminate point is different from the double fixed point.
Case II-B-2 : the indeterminate point coincides with the double fixed point.

The normal form of case II-A-1 is same as the case I-A. There is a restriction on the parameters. Let $\gamma \in \mathbb{C} \setminus \{0, 1\}$ denote the multiplier at one of the fixed point of the Möbius transformation. The parameters
in the normal form are given by $\lambda_1 = \frac{\gamma}{1-\gamma}$, $\lambda_2 = \frac{1}{\gamma-1}$, and $\lambda_3 = 0$.

(II-A-1) \[
\begin{align*}
x_1 &= x + \frac{\gamma}{1-\gamma}x^2 + \frac{\gamma}{\gamma-1}xy \\
y_1 &= y + \frac{1}{1-\gamma}xy + \frac{1}{\gamma-1}y^2.
\end{align*}
\]

(II-A-2) \[
\begin{align*}
x_1 &= x + \lambda x^2 \\
y_1 &= y + (\lambda + 1)xy.
\end{align*}
\]

Here, the parameter (translation number) $\lambda$ is given by $\lambda = \frac{\gamma}{1-\gamma}$, for multiplier $\gamma \in \mathbb{C} \setminus \{0, 1\}$ of the Möbius transformation at the indeterminate fixed point. Note that the cases $\lambda = 0$ and $\lambda = -1$ are omitted here. These cases will be treated as cases III-A-2 and III-B-1 below.

The case II-B-1 corresponds to the exceptional case of I-B with $\lambda = 0$.

(II-B-1) \[
\begin{align*}
x_1 &= x + xy \\
y_1 &= y + xy + y^2.
\end{align*}
\]

(II-B-2) \[
\begin{align*}
x_1 &= x + x^2 \\
y_1 &= y + x^2 + xy.
\end{align*}
\]

(II-C) \[
\begin{align*}
x_1 &= x + x^2 \\
y_1 &= y + xy.
\end{align*}
\]

In the case III, the induced map $[H]$ yields a constant function on the Riemann sphere. We have the following sub-cases according to the common factors of the components of $H$.

CASE III-A : the components $f(x, y)$ and $g(x, y)$ have two mutually prime common factors.

CASE III-B : the components $f(x, y)$ and $g(x, y)$ have a double common factor.

The common factor defines the indeterminate points of the induced map $[H]$. The value of the constant function $[H]$ is defined except at these indeterminate points. Let $v([H])$ denote the value. Taking these points into considerations, we have following sub-cases.

CASE III-A-1 : $v([H])$ is different from the indeterminate points.

CASE III-A-2 : $v([H])$ coincides with one of the indeterminate points.

CASE III-B-1 : $v([H])$ is different from the double indeterminate point.
CASE III-B-2: $v([H])$ coincides with the double indeterminate point.

The case III-A-1 falls into the normal form I-A with excepted parameters $\lambda_1 = \lambda_2 = 0$, and a simpler normal form is given by

$$\begin{align*}
(\text{III-A-1}) & \quad \begin{cases}
x_1 &= x \\
y_1 &= y + xy + y^2.
\end{cases}
\end{align*}$$

The normal form for case III-A-2 is obtained by setting $\lambda = 0$ in II-A-2.

$$\begin{align*}
(\text{III-A-2}) & \quad \begin{cases}
x_1 &= x \\
y_1 &= y + xy.
\end{cases}
\end{align*}$$

The normal form for case III-B-1 is obtained by setting $\lambda = -1$ in II-A-2.

$$\begin{align*}
(\text{III-B-1}) & \quad \begin{cases}
x_1 &= x \\
y_1 &= y + y^2.
\end{cases}
\end{align*}$$

$$\begin{align*}
(\text{III-B-2}) & \quad \begin{cases}
x_1 &= x \\
y_1 &= y + x^2.
\end{cases}
\end{align*}$$

Finally, the case IV has the 2-jet normal form

$$\begin{align*}
(\text{IV}) & \quad \begin{cases}
x_1 &= x \\
y_1 &= y.
\end{cases}
\end{align*}$$

Here, we note the correspondence between our classification of 2-jet normal forms of parabolic analytic transformations and that of Ueda's classification[2].

<table>
<thead>
<tr>
<th>Ueda's notation</th>
<th>our classification</th>
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<td>$N_1(\lambda_1, \lambda_2, \lambda_3)$</td>
<td>I-A,</td>
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</tr>
<tr>
<td>$N_{2,1}(\lambda)$</td>
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<td>$N_4$</td>
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<td>IV</td>
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2. Pseudo-parabolic fixed points

In this section, we consider the case I-B. In this case, the induced map $[H]$ has a simple fixed point and a double fixed point. The translation number $\lambda$ in the normal form I-B is related to the simple fixed point. We call a fixed point of type I-B a pseudo-parabolic fixed point. We are interested in the double fixed point of $[H]$. In order to study the behavior of the local analytic transformation in the neighborhood of the pseudo-parabolic fixed point, we consider the blow-up $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ of $\mathbb{C}^2$ at $O$. We denote the exceptional curve by $\Theta = \pi^{-1}(O) \simeq \mathbb{C}$. Let $V$ be the domain of definition of the transformation $T$ and let $\overline{V} = \pi^{-1}(V)$. The transformation induces an analytic transformation $\overline{T} : \overline{V} \to \mathbb{C}^2$. As $dT(O) = id$, all points of the exceptional curve are fixed points of $\overline{T}$.

Let us try a blow-up in our case I-B. The $x$-axis direction, $\{y = 0\}$, corresponds to the simple fixed point of $[H]$, and is related to the translation number $\lambda$. To see this, we may try a blow-up with $t = \frac{y}{x}$. We obtain the following local analytic transformation.

\[
\begin{aligned}
x_1 &= x + (\lambda + t)x^2 + \cdots \\
t_1 &= t + tx + \cdots.
\end{aligned}
\]

(2.3)

The $y$-axis direction, $\{x = 0\}$, corresponds to the parabolic fixed point of $[H]$. We try a blow-up with $u = \frac{x}{y}$ and obtain the following.

\[
\begin{aligned}
y_1 &= y + (1 + (\lambda + 1)u)y^2 + \cdots \\
u_1 &= u - u^2y + \cdots.
\end{aligned}
\]

(2.4)

Local analytic transformations arising from such a blow-up leaves the exceptional curve invariant, and all the points in the exceptional curve are fixed points. By taking a system of local coordinates around the point in the exceptional curve, we can assume, in general, that the local analytic transformation is of the following form.

\[
\begin{aligned}
x_1 &= x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\
y_1 &= y + g_1(y)x + g_2(y)x^2 + \cdots.
\end{aligned}
\]

(2.5)

Local analytic transformations of the form (2.5) is called a transformation of class $S_{\nu}$, $\nu = 0, 1, 2, \cdots$ [resp. class $S_{\infty}$] if $g_1(y)$ vanishes at $y = 0$ exactly with order $\nu$ [resp. vanish identically]. For $T \in S_1$, we define the translation number $\lambda$ by

$$\lambda = \frac{f_2(0)}{g_1'(0)}.$$
The translation number \( \lambda \) and the multiplier \( \mu \) of the corresponding simple fixed point of \([H]\) are related by \( \lambda = \frac{1}{\mu - 1} \). The translation number is also a holomorphic invariant in class \( S_1 \).

For \( T \in S \), the order of vanishing of \( g_1(y) \) at \( y = 0 \) is invariant under those holomorphic change of coordinates which transforms the transformation of the form (2.5) into the same form.

Let \( T \) be a local analytic transformation, and \( T \in S_1 \). The origin has a basin of attraction if the real part of the translation number is positive. We call this basin of attraction a parabolic basin of the parabolic fixed point.

Note that (2.3) is of class \( S_1 \) and its translation number is \( \lambda \). The transformation for the double fixed point (2.4) is of class \( S_2 \), which shall be discussed in the following section.

3. Weakly-parabolic basin

In this section, we consider a local analytic transformation \( T \in S_2 \) given by

\[
\begin{align*}
{x_1} &= x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\
y_1 &= y + g_1(y)x + g_2(y)x^2 + \cdots,
\end{align*}
\]

where \( g_1(0) = 0 \), \( g_1'(0) = 0 \), and \( g_1''(0) \neq 0 \).

**Theorem 3.1** If \( f_2(0) \neq 0 \), local analytic transformation (3.1) has a non-empty basin of attraction.

We call this attractive basin a weakly-parabolic basin. As a preliminary, we try to simplify the transformation by local change of coordinates.

**Proposition 3.2** For any \( \delta \in \mathbb{C} \), by a change of coordinates \( S_\alpha : (X, Y) \mapsto (x, y) \) of the form

\[
\begin{align*}
x &= \alpha(Y)X \\
y &= Y,
\end{align*}
\]

where \( \alpha(Y) \) is an analytic function of \( Y \), transformation (3.1) can be transformed into the form

\[
\begin{align*}
X_1 &= X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\
Y_1 &= Y + G_1(Y)X + G_2(Y)X^2 + \cdots,
\end{align*}
\]

with \( F_2(Y) = 1 + \delta Y + \cdots \), \( G_1(0) = G_1'(0) = 0 \), and \( G_1''(0) \neq 0 \).
The function $\alpha(Y)$ must satisfy the differential equation

\[ f_2(Y)\alpha(Y) - g_1(Y)\alpha'(Y) = F_2(Y), \]

with $\alpha(0) \neq 0$. Let

\[ a_0 = \frac{1}{f_2(0)}, \]

\[ a_1 = \frac{1}{f_2(0)}(\delta - a_0 f_2'(0)) = \frac{1}{f_2(0)}(\delta - \frac{f_2'(0)}{f_2(0)}) \]

and choose the analytic function $\alpha(Y)$ as, for example,

\[ \alpha(Y) = a_0 + a_1 Y. \]

We obtain the desired change of coordinates of the proposition. As $\alpha(0) = a_0 \neq 0$, the conditions for $G_1(Y)$ are satisfied.

Especially, as we have $G_1''(0) = f_2(0)g_1''(0)$, we can take $\delta = G_1''(0)/2 = f_2(0)g_1''(0)/2$ to be used in the following proposition.

**Proposition 3.3** Assume $T \in S_2$ and $f_2(y) = 1 + \delta y + O(y^2)$, with $\delta = \frac{g_1''(0)}{2}$. By a change of coordinates $S_\beta : (X, Y) \mapsto (x, y)$ of the form

\[ \begin{cases} 
  x & = X \\
  y & = \beta(Y),
\end{cases} \]

with $\beta(0) = 0$, $\beta'(0) \neq 0$, $T$ can be transformed into $\tilde{T} : (X, Y) \mapsto (X_1, Y_1)$,

\[ \begin{cases} 
  X_1 & = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\
  Y_1 & = Y + G_1(Y)X + G_2(Y)X^2 + \cdots
\end{cases} \]
with $F_2(Y) = 1 + Y + O(Y^2)$ and $G_1(Y) = Y^2 + O(Y^3)$.

**Proof.** Compare both sides of $T \circ S_\beta = S_\beta \circ \overline{T}$ as power series in $X$ and obtain

$$f_2(\beta(Y)) = F_2(Y), \quad g_1(\beta(Y)) = \beta'(Y)G_1(Y).$$

Let $\beta(Y) = \frac{2}{g_1(0)}Y$, for example, to get $G_1(Y) = Y^2 + O(Y^3)$.

Note that, here, generally, a term of order 3 cannot be suppressed by an analytic change of coordinates. We have, also,

$$F_2(Y) = f_2(\beta(Y)) = 1 + Y + O(Y^2).$$

**Proposition 3.4** Let $T: (x, y) \mapsto (x_1, y_1)$ be a local analytic transformation of the form

$$x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots,$$

$$y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots,$$

and let $S: (X, Y) \mapsto (x, y)$ be a change of local coordinates of the form

$$x = \alpha_1(Y)X + \alpha_2(Y)X^2 + \alpha_3(Y)X^3 + \cdots,$$

$$y = \beta_0(Y) + \beta_1(Y)X + \beta_2(Y)X^2 + \cdots.$$

Let $\overline{T}: (X, Y) \mapsto (X_1, Y_1)$ be the transformation given by $\overline{T} = S^{-1} \circ T \circ S$, with

$$X_1 = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots,$$

$$Y_1 = Y + G_1(Y)X + G_2(Y)X^2 + \cdots.$$

Then we have the followings.

$$G_1(Y) = \frac{\alpha_1(Y)}{\beta_0'(Y)}g_1(\beta_0(Y))$$

and

$$F_2(Y) = \alpha_1(Y)f_2(\beta_0(Y)) - \frac{\alpha_1'(Y)}{\beta_0(Y)}g_1(\beta_0(Y)).$$

**Proof.** These are verified by an immediate computation.

**Proposition 3.5** Assume $T \in S_2$ is of the form (3.9) with $f_2(y) = 1 + y + O(y^2)$ and $g_1(y) = y^2 + O(y^3)$. By a local change of coordinates $S$ of the form (3.10), the transformation $T$ can be transformed into $\overline{T}$ of the form (3.11) with $F_2(Y) = f_2(Y)$, $G_1(Y) = g_1(Y)$ and $G_2(Y) = 0$. 
\textbf{Proof} We set $\alpha_1(Y) = 1$ and $\beta_0(Y) = Y$. Then proposition 3.4 guarantees that $G_1(Y) = g_1(Y)$ and $F_2(Y) = f_2(Y)$. Compute $S \circ \overline{T}$ and $T \circ S$ to compare the coefficients of $X^2$ in $y_1$. We get

$$G_2(Y) = g_2(Y) + \beta_1(Y)(g'_1(Y) - f_2(Y)) + g_1(Y)(\alpha_2(Y) - \beta'_1(Y)).$$

Hence, if we set

$$\beta_1(Y) = \frac{g_2(Y)}{f_2(Y) - g'_1(Y)}$$

and

$$\alpha_2(Y) = \beta'_1(Y),$$

we get $G_2(Y) = 0$. As $f_2(Y) = 1 + Y + O(Y^2)$ and $g'_1(Y) = O(Y)$, $\beta_1(Y)$ is analytic near the origin.

\section{Proof of theorem 3.1}

By propositions in the previous section, we can assume

$$f_2(y) = 1 + y + O(y^2),$$

$$g_1(y) = y^2 + O(y^3),$$

and

$$g_2(y) = 0$$

to prove theorem 3.1. Then, the transformation $T : (x, y) \mapsto (x_1, y_1)$, $T \in S_2$, takes the following form

$$\begin{align*}
\{ & x_1 = x + (1 + y)x^2 + O(y^2x^2) + O(x^3) \\
& y_1 = y + y^2x + O(y^3x) + O(x^3),
\end{align*}$$

(4.1)

where $O(\varphi(x, y))$ implies some analytic function, say $\psi(x, y)$, which can be written as $\psi(x, y) = \varphi(x, y)\rho(x, y)$ for some analytic function $\rho(x, y)$ in a neighborhood of the origin.

As we are interested in the behavior of the transformation in the $y$-axis direction near the origin, let us blow-up the origin along the $y$-axis. More precisely, we change the coordinates by

$$u = \frac{x}{y}, \quad v = y$$

(4.2)

into new coordinates $(u, v)$. The origin $(0, 0)$ of $(x, y)$-coordinates corresponds to the exceptional curve $\overline{\mathbb{C}} \times \{0\}$ in the $(u, v)$-coordinates.
In the \((u, v)\)-coordinates, (4.1) takes the form

\[
\begin{align*}
(u_1 &= u + vu^2 + O(v^3u^2) + O(v^2u^3) \\
v_1 &= v + v^3u + O(v^4u) + O(v^3u^3).
\end{align*}
\]

Let us take a new system of coordinates defined by

\[
z = \frac{1}{u}, \quad w = \frac{1}{v}.
\]

Then (4.3) is transformed into the form

\[
\begin{align*}
z_1 &= z - \frac{1}{w}h_1(z, w) \\
w_1 &= w - \frac{1}{zw}h_2(z, w),
\end{align*}
\]

where \(h_1(z, w) = 1 + O(\frac{1}{zw}) + O(\frac{1}{w^2})\) and \(h_2(z, w) = 1 + O(\frac{1}{z^2}) + O(\frac{1}{w})\).

We regard (4.5) as a transformation near \((\infty, \infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}\).

Take constants \(\theta_0, \theta_1, \theta_2\) such that

\[0 < \theta_0 < \frac{1}{8}\pi, \quad 0 < \theta_2 < \frac{1}{8}\theta_0, \quad \text{and} \quad \theta_0 + \theta_2 < \theta_1 < \frac{5}{4}\theta_0 - \theta_2.\]

Note that \(0 < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3}\) holds.

Choose \(r_1\) and \(r_2\) such that \(\frac{3}{4} < r_1 < 1 < r_2 < \frac{5}{4}\) and let

\[
\Omega = \{z \in \mathbb{C} \mid |\arg z| < \theta_2, r_1 < |z| < r_2\}.
\]

For \(R_1, R_2 > 0\), let

\[
U = \{z \in \mathbb{C} \mid |\arg(-z)| < \theta_1, \text{Re} z < -R_1\}
\]

and

\[
V = \{w \in \mathbb{C} \mid |\arg w| < \theta_0, \text{Re} w > R_2\}.
\]

Choose sufficiently large \(R_1\) and \(R_2\) such that

\[
h_1(z, w) \in \Omega \quad \text{and} \quad h_2(z, w) \in \Omega
\]

holds for all \((z, w) \in U \times V\), and that

\[r_2 < R_1R_2^2 \sin\left(\frac{\theta_0}{2}\right).\]

Let \(\Phi : (z, w) \mapsto (z_1, w_1)\) denote the transformation (4.5) defined near \((\infty, \infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}\).

**Proposition 4.1** If \((z, w) \in U \times V\), then \(\Phi(z, w) = (z_1, w_1) \in U \times V\), \(\text{Re} z_1 < \text{Re} z\), and \(\text{Re} w_1 > \text{Re} w\).
PROOF Let \((z, w) \in U \times V\). Then
\[
|\arg\left(\frac{1}{w}h_1(z, w)\right)| < \theta_0 + \theta_2 < \theta_1
\]
and
\[
\text{Re}\left(\frac{1}{w}h_1(z, w)\right) > 0.
\]
Hence \(z_1 \in U\) and \(\text{Re} \ z_1 < \text{Re} \ z\) follow. Now, let \(\theta = \arg w\). Then \(-\theta_0 < \theta < \theta_0\). Note that
\[
|\arg\left(-\frac{1}{zw}h_2(z, w)\right)| < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3}
\]
and
\[
\text{Re}\left(-\frac{1}{zw}h_2(z, w)\right) > 0.
\]
First, consider the case where \(\frac{\theta_0}{2} < \theta < \theta_0\). In this case, we have
\[
\arg\left(-\frac{1}{zw}h_2(z, w)\right) < -\theta + \theta_1 + \theta_2 < \theta_0.
\]
So, we have \(\arg w_1 < \theta_0\) and \(\text{Re} \ w_1 > w > R_2\). On the other hand,
\[
|w_1 - w| = \left| -\frac{1}{zw}h_2(z, w) \right| < \frac{r_2}{R_1R_2} < R_2 \sin \frac{\theta_0}{2}.
\]
Hence \(w_1 \in V\) in this case.

Similarly, if \(-\theta_0 < \theta < -\frac{\theta_0}{2}\), we have \(w_1 \in V\).

Next, if \(|\theta| \leq \frac{\theta_0}{2}\), we have
\[
\text{Re}\left(-\frac{1}{zw}h_2(z, w)\right) > 0 \quad \text{and} \quad |w_1 - w| < R_2 \sin \frac{\theta_0}{2},
\]
which imply \(w_1 \in V\) and \(\text{Re} \ w_1 > \text{Re} \ w\). Thus proposition 4.1 is proved.

Theorem 3.1 is a corollary of this proposition.

References
