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On Termination of One-Rule String Rewriting Systems

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Let Σ be a finite alphabet. The free monoid and the free semigroup generated by Σ are denoted by Σ^* and Σ^+, respectively. The length of a word x in Σ^* is denoted by $|x|$. For $x, y \in \Sigma^*$, we set $\mathrm{OVL}(x, y) = \{z \in \Sigma^+ | \ x = uz, \ y = zv$ for some $u, v \in \Sigma^+\}$. A *rewriting system* R on Σ is a subset of $\Sigma^* \times \Sigma^*$. An element (l, r) in R is denoted by $l \to r$. If R contains only one element, R is said to be a *one-rule rewriting system*. A *single step reduction relation* $\to$ induced by R is the following relation on Σ^*: For any $x, y \in \Sigma^*$, $x \to y$ if and only if there exists $(l, r) \in R$ such that $x = ulv$, $y = urv$ for some $u, v \in \Sigma^*$. $\to^*$ is the reflexive and transitive closure of $\to$.

A rewriting system R is said to be *confluent* if for any $w, x, y \in \Sigma^*$, $w \to^* x$ and $w \to^* y$ imply $x \to^* z$ and $y \to^* z$ for some $z \in \Sigma^*$. R is *terminating* (or *noetherian*) if there is no infinite sequence $x_1, x_2, \cdots$ such that $x_1 \to x_2 \to \cdots$. A confluent and terminating rewriting system is said to be *complete*.

It is not known whether the completeness is decidable for one-rule rewriting systems. Let $R = \{l \to r\}$ be a one-rule rewriting system. If $r \in \Sigma^* l \Sigma^*$ then R is always non-terminating. If $|l| \geq |r|$ and $l \neq r$ then R is always terminating.

Result 1 [3] *It is decidable whether or not a one-rule rewriting system is confluen.*

Result 2 [2] *For a confluent one-rule rewriting system* $R = \{l \to r\}$ *with* $|l| < |r|$, *we can effectively construct a rewriting system* $R' = \{l' \to r'\}$ *such that:*

(1) $|l'| < |r'|$ *and* $\mathrm{OVL}(l', l') = \emptyset$.

(2) R' *is terminating if and only if* R *is terminating.*

Hence the completeness problem for one-rule systems is reduced to the termination problem for one-rule systems $R = \{l \to r\}$ with $\mathrm{OVL}(l, l) = \emptyset$. It is not difficult to see that if $\mathrm{OVL}(r, l) = \emptyset$ or $\mathrm{OVL}(l, r) = \emptyset$ then R is terminating. In this note, we consider the case where $\mathrm{OVL}(r, l) = \{p\}$, a singleton.

For each $s \in \mathrm{OVL}(l, r)$, we determine $\bar{s} \in \Sigma^*$ by $l = \bar{s}s$. The decidability of the terminating problem for such one-rule systems is given as follows.

Theorem 1. *Let $R = \{l \to r\}$ be a one-rule rewriting system such that $\mathrm{OVL}(l, l) = \emptyset$ and $\mathrm{OVL}(r, l) = \{p\}$. Let $l = px$, $r = y\bar{s}_k \cdots \bar{s}_1 p$, where $s_1, \ldots, s_k \in \mathrm{OVL}(l, r)$ and $y \notin \Sigma^*\bar{s}$ for any $s \in \mathrm{OVL}(l, r)$.*

(1) *If there is a reduction of length $|r|^2$ starting with $(y\bar{s}_k \cdots \bar{s}_1)^3$ then R is non-terminating.*

(2) *Assume that the maximal length of reductions starting with $(y\bar{s}_k \cdots \bar{s}_1)^3$ is N with $N < |r|^2$. If $|x| > |y|$ and there is a reduction of length $2N+1$ starting with $(y\bar{s}_k \cdots \bar{s}_1)^4$ then R is non-terminating, otherwise, R is terminating.*

The exact characterization of non-terminating one-rule systems is given as follows.

Theorem 2. *Let $R = \{l \to r\}$ be a one-rule rewriting system such that $\mathrm{OVL}(l, l) = \emptyset$ and $\mathrm{OVL}(r, l) = \{p\}$. Then R is non-terminating if and only if one of the following conditions is satisfied.*
(k, m, n are positive integers. $x, y, z, w \in \Sigma^+$ and $u, v \in \Sigma^*$.)

(1) $l \in \Sigma^* r \Sigma^*$.

(2) $r = s_k u \bar{s}_k \cdots \bar{s}_1 p$, $\quad s_1, \ldots, s_k \in \mathrm{OVL}(l, r) \cap (xu)^* x$.

(3) $l = p(ux)^n$, $\quad r = (xu)^{n+m} \bar{s}_k \cdots \bar{s}_1 p$, $\quad s_1, \ldots, s_k \in \mathrm{OVL}(l, r) \cap (xu)^* x$.

(4) $l = p(ux)^n$, $\quad r = (xu)^{n+m} \bar{s}_k \cdots \bar{s}_1 p$, $\quad s_1, \ldots, s_k \in \mathrm{OVL}(l, r)$,
$s_1, \ldots, s_{j-1} \in (xu)^* x$, $\quad s_j \in (xu)^i x$, $\quad 1 \le j \le k$, $\quad 1 \le i \le 2m$.

(5) $l = pxy$, $\quad r = y(xyz)^m x \bar{s}_k \cdots \bar{s}_1 p$,
$s_1, \ldots, s_k \in \mathrm{OVL}(l, r)$, $\quad s_1 = y(xyz)^m$, $\quad xyz = wxy$.

(6) $l = xy((zx^{mk}y)^{m-1}zy)^{n+1}$, $\quad r = y((zx^{mk}y)^{m-1}zy)^{n+1}zx^{mk}y$

(7) $l = zxyxv\,(z^{2m+n+1}xyxv)^m x$, $\quad pu = zxyxv$,
$r = xyxv\,(z^{2m+n+1}xyxv)^m xuz^{2m+n}p$.

(8) $l = pu((pu)^k z)^{2m+n} pu(((pu)^k z)^{2m+n} p)^{m-1} x$, $\quad pu = zxyxv$.
$r = xyxvz\,((pu)^k z)^{2m+n-1} pu(((pu)^k z)^{2m+n} p)^{m-1} xu((pu)^k z)^{2m+n} p$.

(9) $l = zx(yx)^{k-1}(yz^{m+n+1}x(yx)^{k-1})^m yx$,
$r = x(yx)^{k-1}(yz^{m+n+1}x(yx)^{k-1})^m yxyz^{m+n+1}x(yx)^{k-1}$.

References

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