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Congruence relations and filters in some variety

島根大・総合理工　近藤通朗　(Michiro KONDO)

1 Introduction

There is a close relation between logics and algebras. For example, the Boolean algebras characterize the classical propositional logic (CPL), that is, any provable formula A in CPL is always evaluated 1 in any Boolean algebra B and, conversely, if A is evaluated 1 in every Boolean algebra then it is provable in CPL. We proved the characterization theorem for other algebras, in [3] for Kleene and in [4] for de Morgan algebras. These characterization problems can be treated in a uniform method by using so called a Lindenbaum-Tarski algebra. The algebra is a quotient algebra of all formulas by provability. It is typical in the corresponding algebras to the logic. Hence to answer the characterization problems it is necessary to consider the typical algebras. The algebras above are in the subclass $B_{n,0}$ of the Berman class which is a subclass of Ockham algebras. Thus it is interesting to think about the properties of the algebras in $B_{n,0}$.

Since the Berman's [1], on the other hand, there are many papers about the Ockham algebras and their congruence relations. As a special Ockham algebra, we consider Boolean algebras and their next fundamental results.

If F is a maximal filter of the Boolean algebra B then the relation Θ_F defined by

$$\Theta_F = \{(x,y)|x \wedge f = y \wedge f \text{ for some } f \in F\}$$

is congruent and the quotient algebra B/Θ_F is isomorphic to the typical Boolean algebra $\{0,1\}$. Since B/Θ_F has only trivial congruence relations ω and ι, the algebra is called simple. Conversely if Θ is the congruence relation on B then the set F_Θ defined by

$$F_\Theta = \{x \in B | (x, 1) \in \Theta\}$$

is a filter and moreover it is maximal.

When we consider the algebras L in $B_{n,0}$, the following questions arise naturally:

(Q1) What are the relations between Θ and Θ_{F_Θ}? Is the set F identified with F_{Θ_F} if F is a maximal filter?

(Q2) If F is a maximal filter, then is the relation Θ_F congruent or maximal congruent? Does the converse holds? Does the maximality of the congruence relation Θ yield the maximality of the filter Θ_F? Does the converse hold?

(Q3) Is the relation Φ_F (defined precisely below) induced by the partition by F a congruence one?

We answer the questions in the present paper.

2 Subclass $B_{n,0}$ of Berman class

In this section we define the subclass $B_{n,0}$ of Ockham algebras, where n is a non-negative integer, and give solutions to the question (Q1). By an Ockham algebra we mean an algebra $(L; \wedge, \vee, N, 0, 1)$ of type (2,2,1,0,0) such that

(1) $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice

(2) The unary operator N satisfies that for any $x, y \in L$,

$N1 = 0$ and $N0 = 1$;

$N(x \wedge y) = Nx \vee Ny$ and $N(x \vee y) = Nx \wedge Ny$;

$N^{2n}x = x$.

In the following we denote the algebra $(L; \wedge, \vee, N, 0, 1)$ simply by L if no confusion arises. The operator N^k is defined for any non-negative integer k recursively:

$N^0x = x$ and $N^{k+1}x = N(N^kx)$.

A non-empty subset F of L is called a filter when it satisfies the conditions:

(F1) $x, y \in F$ imply $x \wedge y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A filter F is called proper when it is a proper subset of L. By a maximal filter F, we mean the proper filter F such that $F \subseteq G$ implies $F = G$ for any proper filter G. A proper filter F is called prime if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for every $x, y \in L$.

The next proposition is well known, so we omit the proof.

Proposition 1 *Let F be a maximal filter and $x \notin F$, then there is an element $u \in F$ such that $x \wedge u = 0$.*

A relation Θ on L is called a congruence relation if it is an equivalence relation and has a substitution property, that is,

$$(x, y), (a, b) \in \Theta \text{ imply } (x \wedge a, y \wedge b), (x \vee a, y \vee b), \text{ and } (Nx, Ny) \in \Theta.$$

We shall define a relation Θ_F for F a filter of L and a filter F_Θ for a congruence relation Θ as follows:

$$\Theta_F = \{(x, y) | \exists f \in F\ \forall k \geq 0 (N^k x \wedge f = N^k y \wedge f)\}$$
$$F_\Theta = \{x \in L | (x, 1) \in \Theta\}$$

It is clear that Θ_F is an equivalence relation on L and F_Θ is a filter of L. First of all we consider the relation between Θ_{F_Θ} and Θ.

Lemma 1 $\Theta_{F_\Theta} \subseteq \Theta$.

Proof. Suppose that $(x, y) \in \Theta_{F_\Theta}$. There is an element f in F_Θ such that $N^k x \wedge f = N^k y \wedge f$ for every $k \geq 0$. That is,

$$x \wedge f = y \wedge f,$$
$$Nx \wedge f = Ny \wedge f,$$
$$\dots$$
$$N^{2n-1} x \wedge f = N^{2n-1} y \wedge f.$$

Since $(f, 1) \in \Theta$ and Θ is the congruence relation, we have that $(x \wedge f, x)$ and $(y \wedge f, y)$ are in Θ and hence that $(x, y) \in \Theta$.

The converse inclusion can be proved when the filter F_Θ is maximal.

Theorem 1 *If F_Θ is a maximal filter, then we have $\Theta_{F_\Theta} = \Theta$.*

Proof. It is sufficient to prove that $\Theta \subseteq \Theta_{F_\Theta}$. By definition of F_Θ, we have in general

$$x \in F_\Theta \text{ iff } y \in F_\Theta$$

$Nx \in F_\Theta$ iff $Ny \in F_\Theta$

$\cdots$

$N^{2n-1}x \in F_\Theta$ iff $N^{2n-1}y \in F_\Theta$.

There are in amount 2^{2n} cases whether $N^k x$ are in F_Θ or not. By the way, if $N^k x \notin F_\Theta$ then there exists $u_k \in F_\Theta$ such that $N^k x \wedge u_k = 0$ because F_Θ is maximal. To describe these facts we introduce a new operator e_k for every k such that

$$(N^k x)^{e_k} = \begin{cases} N^k x & \text{if } N^k x \in F_\Theta \\ u_k & \text{if } N^k x \notin F_\Theta \end{cases}$$

, where $u_k \in F_\Theta$ such that $N^k x \wedge u_k = 0$.

Of course the element u_k is not determined uniquely in general. But this does not prevent our proof. It follows from the definition that $(N^k x)^{e_k}$ is in F_Θ for every k.

If we put $\alpha = \bigwedge_k (N^k x)^{e_k} \wedge \bigwedge_k (N^k y)^{e_k}$, then we have $\alpha \in F_\Theta$ by the definition. For that α it is that $N^k x \wedge \alpha = N^k y \wedge \alpha$. Because, for k such that $N^k x \in F_\Theta$, we have $N^k y \in F_\Theta$ and hence that $(N^k x)^{e_k} = N^k x$ and $(N^k y)^{e_k} = N^k y$. Thus it follows that $N^k x \wedge \alpha = \alpha = N^k y \wedge \alpha$.

If $N^k x \notin F_\Theta$, since $N^k y \notin F_\Theta$, then we have $(N^k x)^{e_k} = u_k$ and $(N^k y)^{e_k} = v_k$ where u_k and v_k are in F_Θ and satisfy $N^k x \wedge u_k = N^k y \wedge v_k = 0$. Thus we have $N^k x \wedge \alpha = 0 = N^k y \wedge \alpha$.

In either case we obtain that $N^k x \wedge \alpha = N^k y \wedge \alpha$ for every k. This means that $(x, y) \in \Theta_{F_\Theta}$. That is $\Theta \subseteq \Theta_{F_\Theta}$ and so that $\Theta = \Theta_{F_\Theta}$.

Next we shall think about the case of filters. That is, we proceed with considerations concerning the relation between a filter F and a filter F_{Θ_F}. It is obvious from the definition that $F_{\Theta_F} \subseteq F$. The next lemma can be established.

Lemma 2 *Let F be a maximal filter of L. Then $F_{\Theta_F} = F$ if and only if the condition holds:*

$x \in F \Longrightarrow N^{2j+1}x \notin F$ and $N^{2j}x \in F$ for every $j < n$.

Proof. ("if" part) It is sufficient to show that $F \subseteq F_{\Theta_F}$. We assume that $x \in F$. Since $N^{2j+1}x \notin F$ and $N^{2j}x \in F$, if we put $\alpha = \bigwedge_{k<2n} (N^k x)^{e_k}$ then

$\alpha \in F$. For that α we have $N^{2j+1}x \wedge \alpha = 0 = N1 \wedge \alpha$ and $N^{2j}x \wedge \alpha = \alpha = 1 \wedge \alpha$. This means that $(x, 1) \in \Theta_F$ and hence that $x \in F_{\Theta_F}$. Thus $F \subseteq F_{\Theta_F}$.

("only if" part) Conversely suppose that $x \in F$. It follows from the assumption that $(x, 1) \in \Theta_F$. There exists an element f in F such that $x \wedge f = 1 \wedge f = f$, $Nx \wedge f = N1 \wedge f = 0$, and so on. This implies that $Nx \notin F, N^2x \in F, \ldots$. Hence we have $N^{2j+1}x \notin F$ and $N^{2j}x \in F$.

3 Maximal congruence relation

In this section we answer the question (Q2) above. In general if there is a partition of a set then we can introduce an equivalence relation on the set. Let F be a filter of L. The set L can be devided into 2^{2n} subsets by F as follows:

$$L_{111\ldots1} = \{x | x \in F, Nx \in F, N^2x \in F, \ldots, N^{2n-1}x \in F\}$$
$$L_{101\ldots1} = \{x | x \in F, Nx \notin F, N^2x \in F, \ldots, N^{2n-1}x \in F\}$$
$$\ldots$$
$$L_{000\ldots0} = \{x | x \notin F, Nx \notin F, N^2x \notin F, \ldots, N^{2n-1}x \notin F\}$$

Thus we can define an equivalence relation Φ_F on L as

$$\Phi_F \ni (x, y) \Longleftrightarrow \exists L_{s_0 s_1 s_2 \ldots s_{2n-1}}(x, y \in L_{s_0 s_1 s_2 \ldots s_{2n-1}})$$

, where $s_k \in \{0, 1\}$.

This means that $(x, y) \in \Phi_F$ if and only if $\forall k(N^k x \in F \Leftrightarrow N^k y \in F)$.

We say Φ_F an induced equivalence relation by F. For that relation we can show the next lemma.

Lemma 3 *If F is a prime filter of L, then the induced relation Φ_F by F is a congruence relation on L.*

Proof. We have to prove that for any $(x, y), (a, b) \in \Phi_F$

(1) $(x \wedge a, y \wedge b) \in \Phi_F$;

(2) $(x \vee a, y \vee b) \in \Phi_F$;

(3) $(Nx, Ny) \in \Phi_F$.

From the fact $N^{2n}x = x$, it is clear that the condition (3) holds. We only show the case of (1).

We simply denote an element $p \in L$ as a sequence of 0 and 1 as follows: $p = p_0 p_1 p_2 \ldots p_{2n-1}$, where p_k is deined by

$$p_k = \begin{cases} 1 & \text{if } N^k p \in F \\ 0 & \text{if } N^k p \notin F \end{cases}$$

.

By definition of Φ_F, we have that

$(p, q) \in \Phi_F$ if and only if $\forall k(p_k = q_k)$.

Hence it is sufficient to show that $\forall k(x \wedge a)_k = (y \wedge b)_k$ when $x_k = y_k$ and $a_k = b_k$ for all k.

Since F is a prime filter, we have that

$(p \wedge q)_k = min\{p_k, q_k\}$ if k is even

$(p \wedge q)_k = max\{p_k, q_k\}$ if k is odd.

Thus if k is even then it follows that

$(x \wedge a)_k = min\{x_k, a_k\} = min\{y_k, b_k\} = (y \wedge b)_k$.

In case of k odd, we also obtain that

$(x \wedge a)_k = max\{x_k, a_k\} = max\{y_k, b_k\} = (y \wedge b)_k$.

Therefore in either case we can conclude that

$\forall k(x \wedge a)_k = (y \wedge b)_k$,

that is, $(x \wedge a, y \wedge b) \in \Phi_F$.

The other case (2) can be proved similarly.

This means that Φ_F is the congruence relation on L.

Corollary 1 *F is the trivial filter (i.e., $F = L$) $\Longleftrightarrow \Phi_F = \iota (= L \times L)$*

Further we can show that

Theorem 2 *If F is a maximal filter, then we have $\Theta_F = \Phi_F$.*

Proof. Suppose that $(x, y) \in \Theta_F$. It follows from definition that there is an element of $f \in F$ such that $\forall k(N^k x \wedge f = N^k y \wedge f)$. We have that $\forall k(N^k x \in F \Leftrightarrow N^k y \in F)$ and hence that $\forall k(x_k = y_k)$. This means that $(x, y) \in \Phi_F$.

Conversely we assume that $(x, y) \in \Phi_F$. We note that, since F is the maximal filter, if $x_k = 0$ and hence $N^k x \notin F$ then there is an element $u_k \in F$

such that $N^k x \wedge u_k = 0$. Thus $x_k = 0$ means that $(N^k x)^{e_k} = u_k$. Now we put the element $\alpha = \bigwedge_k (N^k x)^{e_k} \wedge \bigwedge_k (N^k y)^{e_k}$. Clearly $\alpha \in F$. For that α, if $x_k = 1$ then we have $y_k = 1$ and $N^k x \wedge \alpha = \alpha = N^k y \wedge \alpha$. In case of $x_k = 0$, we get that $y_k = 0$ and $N^k x \wedge \alpha = 0 = N^k y \wedge \alpha$. Therefore there exists $\alpha \in F$ such that $N^k x \wedge \alpha = N^k y \wedge \alpha$ for all k. This yileds to $(x, y) \in \Theta_F$.

Hence the theorem can be proved completely.

In order to prove one of the main theorems of this paper, we need the next lemma which is well known in the theory of Ockham algebras (cf. [2]).

Lemma 4 *Let $L \in B_{n,0}$. L is simple if and only if $K(L) = T_2(L)$ and N is injective, where*

$K(L) = \{0, 1\} \cup \{x | Nx = x\}$ *and* $T_2(L) = \{x | N^2 x = x\}$.

By use of the lemma, the following is established.

Theorem 3 *If F is a maximal filter, then the congruence relation Θ_F is maximal, that is, L/Θ_F is a simple algebra.*

Proof. We use the same notation $x_0 x_1 \ldots x_{2n-1}$ for the element $\mathbf{x} \in L/\Theta_F$ as above. Hence we have that

$$\mathbf{1} = 1010\ldots10, \quad \mathbf{0} = 0101\ldots01,$$

and that

$$\mathbf{x} = \mathbf{y} \text{ if and only if } \forall k (x_k = y_k).$$

When the element $\mathbf{x}$ is denoted by $x_0 x_1 \ldots x_{2n-2} x_{2n-1}$, the elements $N\mathbf{x}$ and $N^2\mathbf{x}$ are

$$N\mathbf{x} = x_1 x_2 \ldots x_{2n-2} x_{2n-1} x_0$$

$$N^2\mathbf{x} = x_2 x_3 \ldots x_{2n-1} x_0 x_1.$$

$$\text{Hence } K(L/\Theta_F) = \{\mathbf{0}, \mathbf{1}\} \cup \{\mathbf{x} | N\mathbf{x} = \mathbf{x}\}$$
$$= \{0101\ldots01, 1010\ldots10, 1111\ldots11, 0000\ldots00\}$$

$$\text{and } T_2(L/\Theta_F) = \{\mathbf{x} | N^2\mathbf{x} = \mathbf{x}\}$$
$$= \{0101\ldots01, 1010\ldots10, 1111\ldots11, 0000\ldots00\}.$$

So we have $K(L/\Theta_F) = T_2(L/\Theta_F)$. It is obvious that N is injective in L/Θ_F. Therefore the algebra L/Θ_F is simple and hence the congruence relation Θ_F is maximal.

Corollary 2 *Let $L \in B_{n,0}$. If F is a maximal filter of L, then the quotient algebra $L/\Theta_F = L/\Phi_F(\in B_{n,0})$ is a simple finite algebra.*

In the above we show that if F is maximal then so Θ_F is and $\Theta_F = \Phi_F$. We can also show the converse.

Theorem 4 *If Θ_F is a maximal congruence relation and $\Theta_F = \Phi_F$, then F is the maximal filter.*

Proof. Suppose that F is not a maximal filter. There is a maximal filter G such that $F \subseteq G$ and $F \neq G$. That is, there exists an element $x \in L$ such that $x \in G$ but $x \notin F$. It follows from $F \subseteq G$ that $\Theta_F \subseteq \Theta_G$ and hence that $\Theta_F = \Theta_G$ or $\Theta_G = \iota$ by Θ_F being maximal. Clearly $\Theta_G \neq \iota$. Hence we have $\Theta_F = \Theta_G$. Since $\Phi_F = \Theta_F$ and G is maximal, it is that $\Theta_G = \Phi_G$ so that $\Phi_F = \Phi_G$. Now consider the element $\mathbf{x}$ in $L/\Phi_F = L/\Phi_G$. Since $x \notin F$, we have $\mathbf{x}$ is $0x_1x_2...x_{2n-1}$ in L/Φ_F. On the other hand, since $x \in G$, the element $\mathbf{x}$ have to be $1x_1x_2...x_{2n-1}$ in L/Φ_G. But this is a contradiction. Therefore the filter F is maximal.

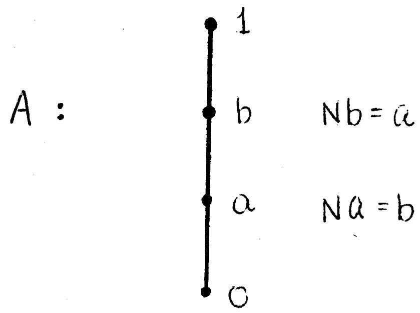
For the congruence relations F_Θ and Θ, we have only the following:

Corollary 3 *If F_Θ is a maximal filter, then the congruence relation Θ is maximal.*

Proof. If F_Θ is maximal, then Θ_{F_Θ} is maximal and $\Theta = \Theta_{F_\Theta}$. Hence we obtain that Θ is a maximal congruence relation.

Unfortunately, the converse does not hold by the following example.

Example: For the algebra $A = \{0, a, b, 1\}$, a congruence relation $\Theta = \{(0,0), (a,a), (b,b), (1,1), (0,a), (a,0), (b,1), (1,b)\}$ is maximal but the filter $F_\Theta = \{b, 1\}$ is not maximal.



4 Filters and induced congruence relations

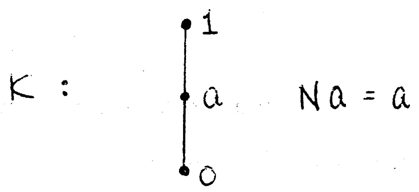
In this section we investigate the property of the induced congruence relation Φ_F by a filter F. In the above we show that if F is a prime filter then Φ_F is a congruence relation. In this case it is a natural question whether the converse holds or not. We answer the question "yes".

Theorem 5 *If Φ_F is a congruence relation on L, then the filter F is prime.*

Proof. Suppose that Φ_F is a congruence relation but the filter F is not prime. There are elements $x, y \in L$ such that $x \vee y \in F$ but $x, y \notin F$. Since Φ_F is congruent, the operations $\wedge, \vee, N$ are closed in the algebra L/Φ_F. When we think about the elements $\mathbf{x} \vee \mathbf{y}, \mathbf{x}$, and $\mathbf{y}$ in L/Φ_F, since $x \vee y \in F$ but $x, y \notin F$, we have $(\mathbf{x} \vee \mathbf{y})_0 = max\{x_0, y_0\} = 1$ but $x_0 = y_0 = 0$ This is a contradiction. Thus F is the prime filter.

Clearly if F is a maximal filter then it is also a prime one. By the theorem, if F is maximal then the congruence relations Θ_F and Φ_F are identified. As in the following, however, the converse does not hold.

Example: Let $K = \{0, a, 1\}$ be the structure as below.



It is obvious that the algebra K is in $B_{1,0}$. If we put $F = \{1\}$, then F is a filter but not maximal. However, as to the congruence relations Θ_F and Φ_F, we have $\Theta_F = \Phi_F (= \omega)$.

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Michiro Kondo
Department of computer and Information Sciences
Shimane University
Matsue Shimane, 690
JAPAN