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Compositions of nondeterministic automata†

B. Imreh
Dept. of Informatics, József Attila University
Árpád tér 2, H-6720 Szeged, Hungary

1. Introduction

In the theory of finite automata it is a central problem how we can built difficult automata from simpler ones. For producing new automata from given ones, different compositions and representations are used. One of the main representations is the isomorphism. An important area of the researches is to characterize those systems of automata which are isomorphically complete, i.e., every automaton is an isomorphic image of a subautomaton of a product from them. The first characterization of the isomorphically complete systems with respect to the most general composition was given by V. M. Glushkov in [7]. As a generalization of the cascade product or serial composition, a product hierarchy, the α_i-product, $i = 0, 1, \ldots$, was introduced by F. Gécseg in [3]. Regarding the α_i-products, the descriptions of the isomorphically complete systems were presented in [9], furthermore, the characterizations of the isomorphically complete systems for some special classes of automata were studied in the works [2],[10],[11], and [12]. A systematic summary of the results on this product hierarchy is given in the monograph [4]. Another family of products, the ν_i-product, $i = 1, 2, \ldots$, was introduced in [1] where the description of the isomorphically complete systems with respect to this family is presented as well. A further composition, the cube-product, was introduced in [13], and it is proved that this composition is equivalent to the general product with respect to the isomorphically complete systems. All of the investigations mentioned above concern deterministic automata. On the other hand, as a consequence of parallel computation, the importance of nondeterministic automata is increasing which gives a motivation to study the isomorphically complete systems of nondeterministic automata. The first description of the isomorphically complete systems of nondeterministic automata with respect to the general product was given in [5]. In the work [6], it is proved that the cube-product is equivalent to the general product regarding the isomorphically complete systems for nondeterministic case, too. It is not known whether there exist finite isomorphically complete systems of nondeterministic automata for such weaker products as α_i-product or ν_i-product in nondeterministic case. Here, we

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recall the characterization of the isomorphically complete systems of nondeterministic automata with respect to the general product, and introduce such a special class of nondeterministic automata, namely, the class of definite nondeterministic automata, for which there are finite isomorphically complete systems with respect to the α_0-product.

2. Preliminaries

We recall some basic concepts of relational systems (see [8]), and introduce the necessary notions of automata theory.

By a *nondeterministic automaton* we mean a couple $\mathbf{A} = (X, A)$ where X and A are nonempty finite sets, furthermore, x is realized as a binary relation $x^{\mathbf{A}}$ on A, for all $x \in X$. The elements of A are called also *states.* For any $a \in A$, $x \in X$, we denote the set $\{a' : a' \in A \;\&\; ax^{\mathbf{A}}a'\}$ by $ax^{\mathbf{A}}$. Let $p \in X^*$ and $a \in A$ be an arbitrary word and state, respectively. Then, we define $ap^{\mathbf{A}}$ by $ap^{\mathbf{A}} = \{a\}$ if $p = \lambda$ where λ denotes the empty word of X^* and $ap^{\mathbf{A}} = \cup\{bx^{\mathbf{A}} : b \in aq^{\mathbf{A}}\}$ if $p = qx$ for some $q \in X^*$ and $x \in X$.

Let $\mathbf{A} = (X, A)$ and $\mathbf{B} = (X, B)$ be nondeterministic automata. $\mathbf{B}$ is called a *subautomaton* of $\mathbf{A}$ if $B \subseteq A$ and $x^{\mathbf{B}}$ is the restriction of $x^{\mathbf{A}}$ to B, for all $x \in X$. A mapping μ of A into B is called a *homomorphism* of $\mathbf{A}$ into $\mathbf{B}$ if the equality $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{B}}$ is valid, for all $a \in A$ and $x \in X$. In particular, if μ is an onto mapping, then we say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$. If μ is a one-to-one homomorphism of $\mathbf{A}$ onto $\mathbf{B}$, then we call μ an *isomorphism* of $\mathbf{A}$ onto $\mathbf{B}$. We also say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic.*

Let us consider the nondeterministic automata $\mathbf{A} = (X, A)$, $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \ldots, m$ and let Φ be a family of feed-back functions below

$$\varphi_j : A_1 \times \ldots \times A_m \times X \to X_j, \quad j = 1, \ldots, m.$$

It is said that $\mathbf{A}$ is the *general product* of $\mathbf{A}_j$, $j = 1, \ldots, m$, if the following conditions are satisfied:

(1) $A = \prod_{j=1}^{m} A_j$,

(2) for any pair $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m) \in A$ and $x \in X$, the relation $(a_1, \ldots, a_m)x^{\mathbf{A}}(b_1, \ldots, b_m)$ is valid if and only if $a_j x_j^{\mathbf{A}_j} b_j$ holds with $x_j = \varphi_j(a_1, \ldots, a_m, x)$, for all $j \in \{1, \ldots, m\}$.

For the general product introduced above, we use the notation

$$\mathbf{A} = \prod_{j=1}^{m} \mathbf{A}_j(X, \Phi) .$$

Let Σ be a system of nondeterministic automata. It is said that Σ is *isomorphically complete* with respect to the general product if, for any nondeterministic automaton $\mathbf{B}$, there exist nondeterministic automata $\mathbf{A}_j \in \Sigma$, $j = 1, \ldots, m$, such that $\mathbf{B}$ is isomorphic to a subautomaton of a general product of the automata $\mathbf{A}_j$, $j = 1, \ldots, m$.

Now, we can recall the main result of [5]. Namely, the isomorphically complete systems of nondeterministic automata with respect to the general product can be characterized as follows.

Theorem 1.([5]) *A system Σ of nondeterministic automata is isomorphically complete with respect to the general product if and only if Σ contains (not necessarily distinct) nondeterministic automata $\mathbf{A}_t = (X_t, A_t)$, $t = 1, 2$, for which there exist states $a_t \neq b_t \in A_t$, $t = 1, 2$, and symbols $x_t, y_t, z_t \in X_t$, $t = 1, 2$, such that*

$$\{a_t, b_t\} \subseteq a_t x_t^{\mathbf{A}_t},\ \{a_t, b_t\} \subseteq b_t y_t^{\mathbf{A}_t},\ t = 1, 2,$$
$$\{a_1, b_1\} \cap a_1 z_1^{\mathbf{A}_1} = \{b_1\},\ \{a_2, b_2\} \cap a_2 z_2^{\mathbf{A}_2} = \{a_2\}.$$

Applying another feed-back functions, we can define another compositions. One of them is the cube-product which can be visualized as follows. We consider an n-dimensional hypercube and 2^n nondeterministic automata. The considered automata are taken in the vertices of the hypercube and the input symbol of any vertex automaton depends on the imput symbol of the composition and on the actual states of the neighbour automata of the considered one where the neighbours are determined by the hypercube. The precise definition of this composition can be found in [13] for deterministic case and in [6] for nondeterministic case. In the last work, the isomorphically complete systems of nondeterministic automata were characterized with respect to the cube-product and it turned out that they are the same ones as in the case of the general product. Therefore, the cube-product is equivalent to the general product with respect to the isomorphically complete systems of nondeterministic automata. Since both the general product and the cube-product are strong ones, it is an interesting problem, whether there are finite isomorphically complete systems for weaker compositions. To study this problem, we extend the notion of the α_0-product to nondeterministic automata and investigate a special class of automata.

Let $\mathbf{A} = (X, A)$, $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \ldots, m$, be nondeterministic automata and let Φ be a family of the following feed-back functions

$$\varphi_1 : X \to X_1,$$
$$\varphi_j : A_1 \times \ldots \times A_{j-1} \times X \to X_j,\quad j = 2, \ldots, m.$$

It is said that $\mathbf{A}$ is the α_0-*product* of $\mathbf{A}_j$, $j = 1, \ldots, m$, if the following conditions are satisfied:

(3) $A = \prod_{j=1}^{m} A_j$,

(4) for any pair $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m) \in A$ and $x \in X$, the relation $(a_1, \ldots, a_m)x^{\mathbf{A}}(b_1, \ldots, b_m)$ is valid if and only if $a_j x_j^{\mathbf{A}_j} b_j$ holds with $x_j = \varphi_j(a_1, \ldots, a_{j-1}, x)$, for all $j \in \{1, \ldots, m\}$.

The proof of the following statement can be given in a straightforward way.

Lemma 1. *If a nondeterministic automaton* $\mathbf{A}$ *can be embedded isomorphically into an* α_0*-product of nondeterministic automata* $\mathbf{A}_t$, $t = 1, \ldots, k$, *and* $\mathbf{A}_t$ *can be embedded isomorphically into an* α_0*-product of nondeterministic automata* $\mathbf{B}_{tj}$, $j = 1, \ldots, s_t$, *for all* t, $t = 1, \ldots, k$, *then* $\mathbf{A}$ *can be embedded isomorphically into an* α_0*-product of* $\mathbf{B}_{11}, \ldots, \mathbf{B}_{1s_1}, \mathbf{B}_{21}, \ldots, \mathbf{B}_{2s_2}, \ldots, \mathbf{B}_{k1}, \ldots, \mathbf{B}_{ks_k}$.

Finally, let Θ be an arbitrary class of nondeterministic automata. It is said that a system Σ of nondeterministic automata is *isomorphically complete* for Θ with respect to the α_0-product if, for any nondeterministic automaton $\mathbf{B} \in \Theta$, there exist $\mathbf{A}_j \in \Sigma$, $j = 1, \ldots, m$, such that $\mathbf{B}$ is isomorphic to a subautomaton of an α_0-product of the automata $\mathbf{A}_j$, $j = 1, \ldots, m$.

In what follows, we introduce a special class of nondeterministic automata and show that there exist finite isomorphically complete systems for this special class with respect to the α_0-product.

3. Definite nondeterministic automata

A nondeterministic automaton $\mathbf{A} = (X, A)$ is called *definite* if the following conditions are satisfied by $\mathbf{A}$:

(i) $ax^{\mathbf{A}} \neq \emptyset$, for all $x \in X$ and $a \in A$,

(ii) there exists a nonnegative integer n such that $|p| \geq n$ implies $|Ap^{\mathbf{A}}| = 1$ for any $p \in X^*$ where $Ap^{\mathbf{A}} = \cup\{ap^{\mathbf{A}} : a \in A\}$.

Let us observe that (1) immediately yields $ap^{\mathbf{A}} \neq \emptyset$, for all $a \in A$ and $p \in X^*$. Let us suppose that $|A| \geq 2$. Let us denote the set of all subsets of A with at least two elements by T and define a binary relation on T as follows. For any $U, V \in T$, $U \leq V$ if and only if there exists a word $p \in X^*$ satisfying $Up^{\mathbf{A}} = V$. Obviously, the defined relation is reflexive and transitive, furthermore, it is antisymmetric since $\mathbf{A}$ is definite. Consequently, it is a partial ordering on T. Let $Q = \{V : V \in T \ \& \ A \leq V\}$. Since $A \in Q$, we have $Q \neq \emptyset$. Restricting the introduced relation to Q, we obtain a partial ordering on Q. Let $(Q, \leq)$ denote the corresponding partially ordered set. Then, we have that $p \in X^+$ implies $|Vp^{\mathbf{A}}| = 1$ for any maximal element of $(Q, \leq)$. Indeed, since V is a maximal element of $(Q, \leq)$, we have $|Vp^{\mathbf{A}}| \leq 1$. On the other hand, $|Vp^{\mathbf{A}}| \geq 1$ is valid by (i).

Let $V \in Q$ be an arbitrary maximal element of $(Q, \leq)$ and define the equivalence relation ρ on A as follows: for any $a, b \in A$,

$$a\rho b \text{ if and only if } \{a, b\} \subseteq V \text{ or } a = b.$$

Now, we can define a new nondeterministic automaton $\mathbf{A}' = (X, A/\rho)$ as follows. For any $a \in A \setminus V$ and $x \in X$,

$$ax^{\mathbf{A}'} = \begin{cases} ax^{\mathbf{A}} & \text{if } ax^{\mathbf{A}} \cap V = \emptyset, \\ (ax^{\mathbf{A}} \setminus V) \cup \{V\} & \text{otherwise,} \end{cases}$$

furthermore,

$$Vx^{\mathbf{A}'} = \begin{cases} V & \text{if } ux^{\mathbf{A}} \cap V \neq \emptyset, \\ ux^{\mathbf{A}} & \text{otherwise,} \end{cases}$$

where u is an arbitrarily fixed element of V. We note that the definition is independent of the choice of u. Indeed, $Vx^{\mathbf{A}}$ is singleton, moreover, $ux^{\mathbf{A}} \neq \emptyset$ and $vx^{\mathbf{A}} \neq \emptyset$ from (1), and therefore, $Vx^{\mathbf{A}} = ux^{\mathbf{A}} = vx^{\mathbf{A}}$ for any $v \in V$.

Lemma 2. *If $\mathbf{A} = (X, A)$ is a definite nondeterministic automaton with $|A| \geq 2$, then $\mathbf{A}' = (X, A/\rho)$ is a definite nondeterministic automaton as well.*

Proof. Since $Q \neq \emptyset$, there exists at least one maximal element of $(Q, \leq)$ which is denoted by V. If $V = A$, then the statement is obvious. Now, let us suppose that $V \subset A$. Let us consider the mapping μ of A onto A/ρ defined by

$$a\mu = \begin{cases} a & \text{if } a \notin V, \\ V & \text{otherwise.} \end{cases}$$

We prove that μ is a homomorphism of $\mathbf{A}$ onto $\mathbf{A}'$. For this purpose, let $a \in A$, $x \in X$ be arbitrary elements. We distinguish the following cases.

Case 1: $a \in A \setminus V$ and $ax^{\mathbf{A}} \cap V = \emptyset$. Then, $a\mu = a$ and $ax^{\mathbf{A}}\mu = ax^{\mathbf{A}}$. Therefore, we obtain that $ax^{\mathbf{A}}\mu = ax^{\mathbf{A}} = ax^{\mathbf{A}'} = a\mu x^{\mathbf{A}'}$.

Case 2: $a \in A \setminus V$ and $ax^{\mathbf{A}} \cap V = \{u_1, \ldots, u_r\}$. Then, we have the following equalities: $ax^{\mathbf{A}}\mu = ((ax^{\mathbf{A}} \setminus V) \cup \{u_1, \ldots, u_r\})\mu = (ax^{\mathbf{A}} \setminus V) \cup \{V\} = ax^{\mathbf{A}'} = a\mu x^{\mathbf{A}'}$.

Case 3: $a \in V$ and $ax^{\mathbf{A}} \cap V \neq \emptyset$. Now, $a\mu = V$ and $ax^{\mathbf{A}} = Vx^{\mathbf{A}} = \{u\}$ for some $u \in V$. Thus, $ax^{\mathbf{A}}\mu = \{u\}\mu = V = Vx^{\mathbf{A}'} = a\mu x^{\mathbf{A}'}$.

Case 4: $a \in V$ and $ax^{\mathbf{A}} \cap V = \emptyset$. Then $a\mu = V$ and $ax^{\mathbf{A}} = \{b\}$ for some $b \in A \setminus V$. Thus, $ax^{\mathbf{A}}\mu = \{b\}\mu = \{b\} = Vx^{\mathbf{A}'} = a\mu x^{\mathbf{A}'}$.

Since $\mathbf{A}$ is definite, there is a nonnegative integer n for which $|Ap^{\mathbf{A}}| = 1$, for all $p \in X^*$ with $|p| \geq n$. Now, let $p \in X^*$ be arbitrary word with $|p| \geq n$. Then, $|Ap^{\mathbf{A}}| = 1$. On the other hand, $(Ap^{\mathbf{A}})\mu = (A\mu)p^{\mathbf{A}'} = (A/\rho)p^{\mathbf{A}'}$ since μ is a homomorphism of $\mathbf{A}$ onto $\mathbf{A}'$. Therefore, $(A/\rho)p^{\mathbf{A}'}$ is also singleton, and thus,

$|(A/\rho)p^{\mathbf{A}'}| = 1$ if $|p| \geq n$, i.e., (ii) is valid for $\mathbf{A}'$. Finally, let $a' \in A/\rho$ and $x \in X$ be arbitrary state and input symbol, respectively. Then, there is a state $a \in A$ such that $a\mu = a'$. Since μ is a homomorphism, we have $ax^{\mathbf{A}}\mu = a\mu x^{\mathbf{A}'} = a'x^{\mathbf{A}'}$. Now, let us observe that $ax^{\mathbf{A}} \neq \emptyset$ from (i), and thus, $a'x^{\mathbf{A}'} \neq \emptyset$, yielding the validity of (i) for $\mathbf{A}'$. Consequently, $\mathbf{A}'$ is a definite nondeterministic automaton.

Remark. Let us observe that choosing an arbitrary nonsingleton subset V' of some maximal element of $(Q, \leq)$, the above construction can be applied for V', furthermore, Lemma 2 is valid for the nondeterministic automaton determined by V'.

Theorem 2. *A system Σ of nondeterministic automata is isomorphically complete for the class of definite nondeterministic automata with respect to the α_0-product if Σ contains two (not necessarily distinct) nondeterministic automata $\mathbf{A}_1 = (X_1, A_1)$ and $\mathbf{A}_2 = (X_2, A_2)$ for which there exist states $a_t \neq b_t \in A_t$, $t = 1, 2$, and symbols $x_t, z_t \in X_t$, $t = 1, 2$, such that*

$$\{a_t, b_t\} \subseteq a_t x_t^{\mathbf{A}_t},\ \{a_t, b_t\} \subseteq b_t x_t^{\mathbf{A}_t},\ t = 1, 2,$$
$$\{a_1, b_1\} \cap a_1 z_1^{\mathbf{A}_1} = \{b_1\},\ \{a_2, b_2\} \cap a_2 z_2^{\mathbf{A}_2} = \{a_2\}.$$

Proof. We prove that every nondeterministic automaton can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ. We proceed by induction on number of states which is denoted by n. If $n = 1$, then the statement is obvious. Now, let us consider an arbitrary definite nondeterministic automaton $\mathbf{A} = (X, A)$ having two states. Let us denote the states of $\mathbf{A}$ by a and b. From (i) and (ii) it follows, that either $ax^{\mathbf{A}} = bx^{\mathbf{A}} = \{a\}$ or $ax^{\mathbf{A}} = bx^{\mathbf{A}} = \{b\}$, for all $x \in X$. Let us define the α_0-product $\mathbf{D} = \mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_1 \times \mathbf{A}_2(X, \Phi)$ as follows. For any $(c_1, \ldots, c_5) \in \{(a_1, a_1, b_2, b_1, a_2), (b_1, b_1, a_2, a_1, b_2)\}$ and $x \in X$, let

$$\varphi_1(x) = x_1,$$
$$\varphi_2(c_1, x) = \begin{cases} z_1 & \text{if } c_1 = a_1 \ \&\ ax^{\mathbf{A}} = \{b\}, \\ x_1 & \text{otherwise,} \end{cases}$$
$$\varphi_3(c_1, c_2, x) = \begin{cases} z_2 & \text{if } c_1 = b_1 \ \&\ ax^{\mathbf{A}} = \{b\}, \\ x_2 & \text{otherwise,} \end{cases}$$
$$\varphi_4(c_1, c_2, c_3, x) = \begin{cases} z_1 & \text{if } c_1 = b_1 \ \&\ bx^{\mathbf{A}} = \{a\}, \\ x_1 & \text{otherwise,} \end{cases}$$
$$\varphi_5(c_1, c_2, c_3, c_4, x) = \begin{cases} z_2 & \text{if } c_1 = a_1 \ \&\ bx^{\mathbf{A}} = \{a\}, \\ x_2 & \text{otherwise.} \end{cases}$$

In all the remaining cases, let us define the feed-back functions arbitrarily in accordance with the definition of the α_0-product.

Let us consider the subautomaton of $\mathbf{D}$ determined by the following subset $\{(a_1, a_1, b_2, b_1, a_2), (b_1, b_1, a_2, a_1, b_2)\}$ of $A_1 \times A_1 \times A_2 \times A_1 \times A_2$. It is easy to see that this subautomaton is isomorphic to $\mathbf{A}$ under the isomorphism $\mu(a) = (a_1, a_1, b_2, b_1, a_2)$ and $\mu(b) = (b_1, b_1, a_2, a_1, b_2)$.

We note that from the above proof it follows that any two-state reset automaton can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ.

Now, let $n > 2$ and let us suppose that the statement is valid for any $1 \leq k < n$. Let $\mathbf{A} = (X, A)$ be an arbitrary definite nondeterministic automaton with $|A| = n$. Let us construct the corresponding partially ordered set $(Q, \leq)$ for $\mathbf{A}$.

If $Q = \{A\}$, then $|Ax^{\mathbf{A}}| = 1$ from (i) and the definition of Q, for all $x \in X$, i.e., $\mathbf{A}$ is a reset automaton. On the other hand, it is well-known that any reset automaton can be embedded isomorphically into a quasi-direct product of two-state reset automata. By the observation above, any two-state reset automaton can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ. Therefore, by Lemma 1, we obtain that $\mathbf{A}$ can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ.

Now, let us suppose that $Q \neq \{A\}$. Let U denote an arbitrarily fixed maximal element of $(Q, \leq)$. Then, there are $u \neq v \in U$. Let $V = \{u, v\}$. Without loss of generality, we may assume that $A = \{a_1, \ldots, a_n\}$ and $V = \{a_{n-1}, a_n\}$. Let us construct the corresponding definite nondeterministic automaton $\mathbf{A}'$ according to the preliminaries of Lemma 2.

Then $|A'| = n - 1$, and thus, by our induction hypothesis, $\mathbf{A}'$ can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ. In what follows, we define a nondeterministic automaton $\mathbf{B}$ and prove that $\mathbf{A}$ can be embedded isomorphically into an α_0-product of $\mathbf{A}'$ and $\mathbf{B}$. Furthermore, we show that $\mathbf{B}$ can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ. Then, by Lemma 1, we obtain that $\mathbf{A}$ can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ which completes the proof.

Let $\mathbf{B} = (W, B)$ where $B = \{h, a_{n-1}, a_n\}$, $W = \{w_1, \ldots, w_{10}\}$, and let us define the realization of the symbols as follows. Let

$$hw_1^{\mathbf{B}} = \{h\},\ hw_2^{\mathbf{B}} = B,\ hw_3^{\mathbf{B}} = \{h, a_{n-1}\},\ hw_4^{\mathbf{B}} = \{h, a_n\},\ hw_5^{\mathbf{B}} = \{a_{n-1}\},$$
$$hw_6^{\mathbf{B}} = \{a_n\},\ hw_7^{\mathbf{B}} = \{a_{n-1}, a_n\},\ a_{n-1}w_8^{\mathbf{B}} = a_n w_8^{\mathbf{B}} = \{a_n\},$$
$$a_n w_9^{\mathbf{B}} = a_{n-1}w_9^{\mathbf{B}} = \{a_{n-1}\},\ a_n w_{10}^{\mathbf{B}} = a_{n-1}w_{10}^{\mathbf{B}} = \{h\}.$$

In all the remaining cases, let $bw^{\mathbf{B}} = B$ where $b \in B$ and $w \in W$.

We note that $\mathbf{B}$ is not a definite nondeterministic automaton. Now, let us define the α_0-product $\mathbf{A}' \times \mathbf{B}(X, \Phi)$ as follows. For any $x \in X$ and $1 \leq j \leq n - 2$, let

$$\varphi_1(x) = x,$$

$$\varphi_2(a_j, x) = \begin{cases} w_1 & \text{if } a_j x^{\mathbf{A}} \cap V = \emptyset, \\ w_5 & \text{if } a_j x^{\mathbf{A}} = \{a_{n-1}\}, \\ w_6 & \text{if } a_j x^{\mathbf{A}} = \{a_n\}, \\ w_7 & \text{if } a_j x^{\mathbf{A}} = \{a_{n-1}, a_n\}, \\ w_3 & \text{if } a_j x^{\mathbf{A}} \cap V = \{a_{n-1}\} \ \& \ a_j x^{\mathbf{A}} \nsubseteq V, \\ w_4 & \text{if } a_j x^{\mathbf{A}} \cap V = \{a_n\} \ \& \ a_j x^{\mathbf{A}} \nsubseteq V, \\ w_2 & \text{if } a_j x^{\mathbf{A}} \cap V = \{a_{n-1}, a_n\} \ \& \ a_j x^{\mathbf{A}} \nsubseteq V, \end{cases}$$

furthermore,

$$\varphi_2(\{a_{n-1}, a_n\}, x) = \begin{cases} w_{10} & \text{if } a_n x^{\mathbf{A}} \notin V, \\ w_8 & \text{if } a_n x^{\mathbf{A}} = \{a_n\}, \\ w_9 & \text{if } a_n x^{\mathbf{A}} = \{a_{n-1}\}. \end{cases}$$

Let us consider the one-to-one mapping μ of A into $A/\rho \times \{h, a_{n-1}, a_n\}$ which is defined by

$$\mu(a_j) = \begin{cases} (a_j, h) & \text{if } 1 \le j \le n-2, \\ (\{a_{n-1}, a_n\}, a_{n-1}) & \text{if } j = n-1, \\ (\{a_{n-1}, a_n\}, a_n) & \text{if } j = n. \end{cases}$$

Using the equalities $Vx^{\mathbf{A}} = a_n x^{\mathbf{A}} = a_{n-1} x^{\mathbf{A}}$ and $|Vx^{\mathbf{A}}| = 1$, it is easy to see that μ is an isomorphism of $\mathbf{A}$ into $\mathbf{A}' \times \mathbf{B}(X, \Phi)$.

Finally, we have to prove that $\mathbf{B}$ can be embedded isomorphically into an α_0-product of nondeterministic automata from Σ. For this purpose, let us consider the α_0-product

$$\mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{A}_1 \times (W, \Phi)$$

where the feed-back functions are defined as follows.

$\varphi_1(w_i) = x_2$, $i = 1, \ldots, 10$,
$\varphi_2(a_2, w_1) = z_2$,
$\varphi_3(a_2, a_2, w_3) = z_2$,
$\varphi_4(a_2, a_2, a_2, w_4) = z_2$, $\varphi_4(b_2, b_2, b_2, w_8) = z_2$,
$\varphi_5(b_2, b_2, a_2, b_2, w_8) = z_2$, $\varphi_5(b_2, b_2, a_2, b_2, w_9) = z_2$,
$\varphi_5(b_2, b_2, b_2, a_2, w_8) = z_2$, $\varphi_5(b_2, b_2, b_2, a_2, w_9) = z_2$,
$\varphi_6(b_2, b_2, a_2, b_2, a_2, w_9) = z_2$,
$\varphi_7(a_2, a_2, a_2, a_2, b_2, a_2, w_5) = z_1$,
$\varphi_7(b_2, b_2, b_2, a_2, a_2, b_2, w_9) = z_1$,
$\varphi_8(a_2, a_2, a_2, a_2, b_2, a_2, a_1, w_6) = z_1$,
$\varphi_8(b_2, b_2, a_2, b_2, a_2, a_2, b_1, w_8) = z_1$,

$$\varphi_9(a_2,a_2,a_2,a_2,b_2,a_2,a_1,a_1,w_7)=z_1,$$
$$\varphi_{10}(b_2,b_2,a_2,b_2,a_2,a_2,b_1,a_1,b_1,w_{10})=z_1,$$
$$\varphi_{10}(b_2,b_2,b_2,a_2,a_2,b_2,a_1,b_1,b_1,w_{10})=z_1.$$

In all the remaining cases, let

$$\varphi_i(u_1,\ldots,u_{i-1},w)=\begin{cases} x_2 & \text{if } 2\le i<7,\\ x_1 & \text{if } 7\le i\le 10,\end{cases}$$

where $w\in W$, $u_t\in A_2$ if $1\le t<7$, and $u_t\in A_1$ if $7\le t\le 10$.

Considering the defined α_0-product, it can be seen by an easy computation that the subautomaton determined by the states

$$(b_2,b_2,a_2,b_2,a_2,a_2,b_1,a_1,b_1,a_1),$$

$$(b_2,b_2,b_2,a_2,a_2,b_2,a_1,b_1,b_1,a_1),$$

$$(a_2,a_2,a_2,a_2,b_2,a_2,a_1,a_1,a_1,b_1),$$

is isomorphic to **B** under the isomorphism μ which is defined by

$$\mu(a_{n-1})=(b_2,b_2,a_2,b_2,a_2,a_2,b_1,a_1,b_1,a_1),$$

$$\mu(a_n)=(b_2,b_2,b_2,a_2,a_2,b_2,a_1,b_1,b_1,a_1),$$

$$\mu(h)=(a_2,a_2,a_2,a_2,b_2,a_2,a_1,a_1,a_1,b_1).$$

This completes the proof of Theorem 2.

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