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Note on representations of generalized inverse $*$-semigroups[1]

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Abstract

The Munn representation of an inverse semigroup S, in which the semigroup is represented by isomorphisms between principal ideals of the semilattice $E(S)$, is not always faithful. By introducing a concept of a *presemilattice*, Reilly considered of enlarging the carrier set $E(S)$ of the Munn representation in order to obtain a faithful representation of S as an inverse subsemigroup of a structure resembling the Munn semigroup $T_{E(S)}$.

The purpose of this paper is to obtain a generalization of the Reilly's results for generalized inverse $*$-semigroups.

1 Introduction

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $\quad (x^*)^* = x,$

(ii) $\quad (xy)^* = y^*x^*,$

(iii) $\quad xx^*x = x.$

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if it satisfies $e^* = e$. For any subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$-semigroup. It is called a *locally inverse $*$-semigroup* if, for any $e \in E(S)$, eSe is an inverse subsemigroup of S. If $E(S)$ is a normal band, then S is called a *generalized inverse $*$-semigroup*.

Let S and T be regular $*$-semigroups. A homomorphism $\phi : S \to T$ is called a *$*$-homomorphism* if $(a\phi)^* = a^*\phi$. A congruence σ on S is called a *$*$-congruence* if

[1]This is the abstract and the details will be published elsewhere

$(a\sigma)^* = a^*\sigma$. A $*$-congruence σ on S is said to be ***idempotent-separating*** if $\sigma \subseteq \mathcal{H}$, where $\mathcal{H}$ is one of the Green's relations. Denote the maximum idempotent-separating $*$-congruence on S by μ_S or simply by μ. If μ_S is the identity relation on S, S is called *fundamental*. The following results are well-known, and we use them frequently throughout this paper.

Result 1.1 [2]. *Let S be a regular $*$-semigroup. Then we have the following:*

(1) $E(S) = P(S)^2$;

(2) *for any* $a \in S$ *and* $e \in P(S), a^*ea \in P(S)$;

(3) *each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection;*

(4) $\mu_S = \{(a,b) \in S \times S : a^*ea = b^*eb$ *and* $aea^* = beb^*$ *for all* $e \in P(S)\}$.

For a mapping $\alpha : A \to B$, denote the domain and the range of α by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset C of A, $\alpha|_C$ means the restriction of α to C.

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse $*$-semigroups in [3], [4] and [5]. In this paper, we follow [5]. A non-empty set X with a reflexive and symmetric relation σ is called an *ι-set*, and denoted by $(X;\sigma)$. If σ is transitive, that is, if σ is an equivalence relation on X, $(X;\sigma)$ is called a ***transitive ι-set***.

Let $(X;\sigma)$ be an ι-set. A subset A of X is called an ***ι-single subset*** of $(X;\sigma)$ if it satisfies the following condition:

for any $x \in X$, there exists at most one element $y \in A$ such that $(x,y) \in \sigma$.

We consider the empty set to be an ι-single subset. We remark that if $(X;\sigma)$ is a transitive ι-set, a subset A of X is an ι-single subset if and only if, for $x,y \in A$, $(x,y) \in \sigma$ implies $x = y$. A mapping α in $\mathcal{I}_X$, the symmetric inverse semigroup on X, is called a ***partial one-to-one ι-mapping*** on $(X;\sigma)$ if $d(\alpha), r(\alpha)$ are both ι-single subsets of $(X;\sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α, respectively. Denote the set of all partial one-to-one ι-mappings of $(X;\sigma)$ by $\mathcal{LI}_{(X;\sigma)}$.

For any ι-single subsets A and B of $(X;\sigma)$, define $\theta_{A,B}$ by

$$\theta_{A,B} = \{(a,b) \in A \times B : (a,b) \in \sigma\} = (A \times B) \cap \sigma.$$

Since a subset of an ι-single subset is also an ι-single subset, $\theta_{A,B} \in \mathcal{LI}_{(X;\sigma)}$. For any $\alpha, \beta \in \mathcal{LI}_{(X;\sigma)}$, define $\theta_{\alpha,\beta}$ by $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{(X;\sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{LI}_{(X;\sigma)}$ as follows:

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $(\mathcal{LI}_{(X;\sigma)}, \circ, *)$ by $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$ or simply by $\mathcal{LI}_{(X;\sigma)}$. In this paper, we use $\mathcal{LI}_{(X;\sigma)}$ rather than $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$.

Result 1.2 [5]. *For any ι-set $(X;\sigma)$, $\mathcal{LI}_{(X;\sigma)}$, defined above, is a locally inverse $*$-semigroup. If $(X;\sigma)$ is a transitive ι-set, then $\mathcal{LI}_{(X;\sigma)}$ is a generalized inverse $*$-semigroup. In this case, we denote it by $\mathcal{GI}_{(X;\sigma)}$ instead of $\mathcal{LI}_{(X;\sigma)}$.*

Moreover, if σ is the identity relation on X, then $\mathcal{LI}_{(X;\sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X.

We call $\mathcal{LI}_{(X;\sigma)}$ [$\mathcal{GI}_{(X;\sigma)}$] the *$\iota$-symmetric locally [generalized] inverse $*$-semigroup* on the ι-set [the transitive ι-set] $(X;\sigma)$ with the ***structure sandwich set*** $\mathcal{M}$.

Let S be a regular $*$-semigroup, and define a relation Ω on S as follows:

$$(x,y) \in \Omega \iff \text{there exists } e \in E(S) \text{ such that } x\rho_e = y,$$

where $\rho_a (a \in S)$ is the mapping of Sa^* onto Sa defined by $x\rho_a = xa$.

Result 1.3 [5]. *Let S be a locally inverse $*$-semigroup. For each $a \in S$, let*

$$\rho_a : x \mapsto xa \qquad (x \in d(\rho_a) = Sa^*).$$

Then a mapping

$$\rho : a \mapsto \rho_a$$

is a $*$-monomorphism of S into $\mathcal{LI}_{(S;\Omega)}(\mathcal{M})$.

For a partial groupoid X, if there exist a semilattice Y, a partition $\pi : X \sim \sum\{X_e : e \in Y\}$ of X and mappings $\varphi_{e,f} : X_e \to X_f$ $(e \geq f$ in $Y)$ such that

(1) for any $e \in Y$, $\varphi_{e,e} = 1_{X_e}$,

(2) if $e \geq f \geq g$, then $\varphi_{e,f}\varphi_{f,g} = \varphi_{e,g}$,

(3) for $x \in X_e$, $y \in X_f$, xy is defined in X if and only if $x\varphi_{e,ef} = y\varphi_{f,ef}$, and in this case $xy = x\varphi_{e,ef}$,

then X is called a ***strong π-groupoid*** with mappings $\{\varphi_{e,f} : e, f \in Y, e \geq f\}$, and it is denoted by $X(\pi; Y; \{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong π-groupoid. A subset A of X is called a ***π-singleton subset*** of $X(\pi; Y; \{\varphi_{e,f}\})$, if there exists $e \in Y$ such that

$$|A \cap X_f| = \begin{cases} 1 & \text{if } f \in \langle e \rangle, \\ 0 & \text{otherwise,} \end{cases}$$

$$(A \cap X_f)\varphi_{f,g} = A \cap X_g \quad \text{for any } f, g \in \langle e \rangle \text{ such that } f \geq g,$$

where $\langle e \rangle$ is the principal ideal of Y generated by e. In this case, we sometimes denote the π-singleton subset A by $A(e)$. If $A(e)$ is a π-singleton subset, then $|A \cap X_f| = 1$ for any $f \in \langle e \rangle$. We denote the only one element of $A \cap X_f$ by a_f. We remark that, for any π-singleton subset $A(e)$, $A(e) = \{a_e\varphi_{e,f} : f \in \langle e \rangle\}$. Denote the set of all π-singleton subsets of $X(\pi; Y; \{\varphi_{e,f}\})$ by $\mathcal{X}$.

Two π-singleton subsets $A(e)$ and $B(f)$ are said to be π-*isomorphic* to each other, if there exists an isomorphism $\overline{\alpha} : \langle e \rangle \to \langle f \rangle$ as semilattices. In this case, the mapping $\alpha : A(e) \to B(f)$ defined by $a_g\alpha = b_{g\overline{\alpha}}$ $(g \in \langle e \rangle)$ is called a π-*isomorphism* of $A(e)$ to $B(f)$. It is obvious that α is a bijection of $A(e)$ onto $B(f)$, and hence $\alpha \in \mathcal{I}_X$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong π-groupoid. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{(A(e), B(f)) \in \mathcal{X} \times \mathcal{X} : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)}\}.$$

For $(A(e), B(f)) \in \mathcal{U}$, let $T_{A(e),B(f)}$ be the set of all π-isomorphisms of $A(e)$ onto $B(f)$, and let

$$T_{X(\pi)} = \bigcup_{(A(e),B(f)) \in \mathcal{U}} T_{A(e),B(f)}.$$

For any $\alpha, \beta \in T_{X(\pi)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$d(\theta_{\alpha,\beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\},$$
$$r(\theta_{\alpha,\beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\},$$
$$a\theta_{\alpha,\beta} = b \quad \text{if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}.$$

Then $\theta_{\alpha,\beta} \in T_{X(\pi)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\pi)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\pi)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Then $T_{X(\pi)}(\circ, *)$ is a regular $*$-semigroup. We denote it by $T_{X(\pi)}(\mathcal{M})$.

Result 1.4 [4]. *A regular $*$-semigroup $T_{X(\pi)}(\mathcal{M})$ is a generalized inverse $*$-semigroup whose set of projections is partially isomorphic to X.*

Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E, and let $P_i = P(E_i)$. Then $\pi : P \sim \sum\{P_i : i \in I\}$ is a partition of P. For any $i, j \in I$ $(i \geq j)$, define a mappig $\varphi_{i,j} : P_i \to P_j$ by

$$e\varphi_{i,j} = efe \quad \text{for some (any) } f \in P_j.$$

Then $P(\pi; I; \{\varphi_{i,j}\})$ is a strong π-groupoid.

Result 1.5 [4]. *Let S be a generalized inverse $*$-semigroup. For each $a \in S$, let*

$$\tau_a : e \mapsto a^*ea \quad (e \in d(\tau_a) = P(Sa^*)).$$

Then a mapping $\tau : a \mapsto \tau_a$ is a $*$-homomorphism of S into $T_{P(\pi)}(\mathcal{M})$ such that $\tau \circ \tau^{-1} = \mu$.

A regular $*$-subsemigroup T of a regular $*$-semigroup S is said to be *$\mathcal{P}$-full* if $P(T) = P(S)$.

Result 1.6 [4]. *A generalized inverse $*$-semigroup S is fundamental if and only if it is $*$-isomorphic to a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\pi)}(\mathcal{M})$ on a strong π-groupoid $X(\pi; I; \{\varphi_{i,j}\})$ such that $P(T_{X(\pi)}(\mathcal{M}))$ is partially isomorphic to $P(S)$.*

In § 2, by introducing the concept of partially ordered ϱ-set $(X(\trianglelefteq); \{\phi_x\})$, we construct a fundamental generalized inverse $*$-semigroup $T_{X(\trianglelefteq)}(\mathcal{M})$. Also, we shall see that $T_{X(\trianglelefteq)}(\mathcal{M})$ has similar properties with $T_{X(\pi)}(\mathcal{M})$, where $T_{X(\pi)}(\mathcal{M})$ has been given by T. Imaoka, I. Inata and H. Yokoyama [4]. And we shall show that two concepts, strong π-groupoids and partially ordered ϱ-sets, are equivalent.

In § 3, we shall introduce the notion of ω-set $(X(\preccurlyeq); \sigma)$, and construct a generalized inverse $*$-semigroup $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$. Furthermore, let S be a generalized inverse $*$-semigroup with the set of projections P, we shall make two generalized inverse $*$-semigroups $T_{P(\trianglelefteq)}(\mathcal{M})$ and $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$, where the former is obtained in § 2, and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.

2 Fundamental generalized inverse $*$-semigroups

2.1 $T_{X(\trianglelefteq)}(\mathcal{M})$

Let $X(\trianglelefteq)$ be a partially ordered set and , for each $x \in X$, consider an order-preserving mapping $\phi_x : X \to X$. If a relation $\varrho = \{(x, y) \in X \times X : y\phi_x = x, x\phi_y = y\}$ is an equivalence relation on X such that

(P1) $x \trianglelefteq y \Longrightarrow$ for each $y' \in y\varrho$, there exists $x' \in x\varrho$ such that $x' \trianglelefteq y'$,

(P2) a relation $\leq = \{(x\varrho, y\varrho) \in X/\varrho \times X/\varrho :$ there exists $x' \in x\varrho$ such that $x' \trianglelefteq y\}$ is a partial order and $X/\varrho(\leq)$ is a semilattice,

(P3) $x_1 \trianglelefteq y, x_2 \trianglelefteq y$ and $x_1\varrho \leq x_2\varrho \Longrightarrow x_1 \trianglelefteq x_2$,

then $(X(\trianglelefteq); \{\phi_x\})$ is called a *partially ordered ϱ-set.*

Let $(X(\trianglelefteq); \{\phi_x\})$ be a partially ordered ϱ-set. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{(\langle a\rangle, \langle b\rangle) \in \mathcal{X} \times \mathcal{X} : \langle a\rangle \simeq \langle b\rangle (\textit{order isomorphic})\},$$

where $\mathcal{X}$ is the set of all principal ideals of $(X(\trianglelefteq); \{\phi_x\})$. For $(\langle a\rangle, \langle b\rangle) \in \mathcal{U}$, let $T_{\langle a\rangle,\langle b\rangle}$ be the set of all (order) isomorphisms of $\langle a\rangle$ onto $\langle b\rangle$, and let

$$T_{X(\trianglelefteq)} = \bigcup_{(\langle a\rangle,\langle b\rangle)\in\mathcal{U}} T_{\langle a\rangle,\langle b\rangle}.$$

For any $\alpha, \beta \in T_{X(\trianglelefteq)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \varrho\},$$

where ϱ is defined in $(X(\trianglelefteq); \{\phi_x\})$.

Then $\theta_{\alpha,\beta} \in T_{X(\trianglelefteq)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\trianglelefteq)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\trianglelefteq)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Then it is clear that $T_{X(\trianglelefteq)}(\circ, *)$ is a regular $*$-subsemigroup of the ι-symmetric generalized inverse $*$-semigroup $\mathcal{GI}_{(X;\varrho)}(\mathcal{M})$. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{X(\trianglelefteq)}(\mathcal{M})$.

Let S be a generalized inverse $*$-semigroup and $P = P(S)$. We consider P as a partially ordered set with respect to the natural order. Now, we have the following results.

Theorem 2.1 *A regular $*$-semigroup $T_{X(\trianglelefteq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup whose set of projections is order isomorphic to $X(\trianglelefteq)$.*

Corollary 2.2 *A partially ordered set X is order isomorphic to the set of projections of a generalized inverse $*$-semigroup if and only if it is a partially ordered ϱ-set.*

2.2 Representations

Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E, and let $P_i = P(E_i)$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by

$$f\phi_e = efe.$$

Let $e, f \in P$, define a relation $\trianglelefteq$ on P by

$$e \trianglelefteq f \Longleftrightarrow e = fef,$$

that is, $\trianglelefteq$ is the restriction of natural order on S to P.

Lemma 2.3 *The set $(P(\trianglelefteq); \{\phi_e\})$, defined above, is a partially ordered ϱ-set.*

Now, we can consider the generalized inverse $*$-semigroup $T_{P(\trianglelefteq)}(\mathcal{M})$, where $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha$ and β are order isomorphisms among principal ideals of $(P(\trianglelefteq); \{\phi_e\})\}$.

Lemma 2.4 *For any $a \in S$, $P(Sa)$ $(= P(Sa^*a))$ is a principal ideal of $(P(\trianglelefteq); \{\phi_e\})$.*

For any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by

$$e\tau_a = a^*ea,$$

where $e \in \langle aa^* \rangle$. It follows from [4] that $\tau_a \in T_{S(\trianglelefteq)}$ and $\tau_a^* = \tau_{a^*}$. Moreover, for any $a, b \in S$, $\theta_{\tau_a,\tau_b} = \tau_{a^*abb^*}$. And we have the following theorem.

Theorem 2.5 *Let S be a generalized inverse $*$-semigroup such that $E(S) = E$ and $P(S) = P$. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E and $P_i = P(E_i)$. Denote the restriction of the natural order on S to P by $\trianglelefteq$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by $f\phi_e = efe$. Then $(P(\trianglelefteq); \{\phi_e\})$ is a partially ordered ϱ-set and $T_{P(\trianglelefteq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup.*

Moreover, for any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by $e\tau_a = a^*ea$. Then a mapping $\tau : S \to T_{P(\trianglelefteq)}(\mathcal{M})$ $(a \mapsto \tau_a)$ is a $*$-homomorphism and the kernel of τ is the maximum idempotent-separating $*$-congruence on S.

Now, we have the following theorem.

Theorem 2.6 *A generalized inverse $*$-semigroup S is fundamental if and only if it is $*$-isomorphic to a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\trianglelefteq)}(\mathcal{M})$ on a partially ordered ϱ-set $(X(\trianglelefteq);\{\phi_x\})$ such that $P(T_{X(\trianglelefteq)}(\mathcal{M}))$ is order isomorphic to $P(S)$.*

Denote the sets of all partially ordered ϱ-sets and the set of all strong π-groupoids by $\mathbb{P}$ and $\mathbb{S}$, respectively.

Remark 2.7 *Let $(X(\trianglelefteq);\{\phi_x\})$ be any element of $\mathbb{P}$. For any $x\varrho, y\varrho \in X/\varrho$ $(x\varrho \geq y\varrho)$, define a mapping $\overline{\varphi}_{x\varrho,y\varrho} : X_{x\varrho} \longrightarrow X_{y\varrho}$ by*

$$x'\overline{\varphi}_{x\varrho,y\varrho} = y', \text{ where } y' \in y\varrho \text{ such that } y' \trianglelefteq x'.$$

Moreover, we define a partial product on X as follows:

$$xy = \begin{cases} x\overline{\varphi}_{x\varrho,(x\varrho)(y\varrho)} & \text{if } x\overline{\varphi}_{x\varrho,(x\varrho)(y\varrho)} = y\overline{\varphi}_{y\varrho,(x\varrho)(y\varrho)} \\ undefined & \text{otherwise.} \end{cases}$$

Then $(X(\trianglelefteq);\{\phi_x\})\lambda = X(\pi_\varrho; X/\varrho; \{\overline{\varphi}_{x\varrho,y\varrho}\})$ is a strong π-groupoid, where π_ϱ is the partition of X induced by ϱ.

Conversely, let $X(\pi;Y;\{\varphi_{e,f}\})$ be any element of $\mathbb{S}$. For any $x \in X$, define a mapping $\tilde{\phi}_x : X \longrightarrow X$ by

$$y\tilde{\phi}_x = x\varphi_{e,ef},$$

where $x \in X_e$ and $y \in X_f$. If we define $\blacktriangleleft = \{(x,y) \in X \times X : x\tilde{\phi}_y = x\}$, then $X(\pi;Y;\{\varphi_{e,f}\})\mu = (X(\blacktriangleleft);\{\tilde{\phi}_x\})$ is a partially ordered ϱ-set.

Hence the mappings λ, μ from $\mathbb{P}$ to $\mathbb{S}$ and from $\mathbb{S}$ to $\mathbb{P}$, respectively, are well-defined. Moreover $\mu\lambda = 1_{\mathbb{S}}$, and for any $(X(\trianglelefteq);\{\phi_x\}) \in \mathbb{P}$, if $(X(\trianglelefteq);\{\phi_x\})\lambda\mu = (X(\blacktriangleleft);\{\tilde{\phi}_x\})$, then $\trianglelefteq\, = \,\blacktriangleleft$.

By the above argument, for any $(X(\trianglelefteq);\{\phi_x\})$ in $\mathbb{P}$, without loss of generality, we can consider $(X(\trianglelefteq);\{\phi_x\})$ as a member of $\mathbb{P}\lambda\mu$.

Now, let $X(\pi;Y;\{\varphi_{e,f}\})$ be any element of $\mathbb{S}$. If $X(\pi;Y;\{\varphi_{e,f}\})\mu = (X(\trianglelefteq);\{\phi_x\})$. Then we can construct two generalized inverse $*$-semigroups $T_{X(\pi)}(\mathcal{M})$ and $T_{X(\trianglelefteq)}(\mathcal{M})$. In this case, these two generalized inverse $*$-semigroups are $*$-isomorphic.

3 Extensions of $T_{X(\trianglelefteq)}(\mathcal{M})$

3.1 $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$

By a *pre-order* on a set X we shall mean a reflexive and transitive relation. Let $X(\preccurlyeq)$ be a pre-ordered set and let $\nu = \{(a,b) \in X \times X : a \preccurlyeq b \text{ and } b \preccurlyeq a\}$. Then ν is an equivalence relation on X and X/ν is a partially ordered set with respect to the induced relation

(C1) $$a\nu \trianglelefteq b\nu \text{ if and only if } a \preccurlyeq b.$$

We call $\trianglelefteq$ the ***naturally induced order*** on X/ν from $\preccurlyeq$. Clearly ν is the smallest equivalence relation on X for which (C1) defines a partial order on X/ν. We call ν the ***minimum partial order congruence*** (mpo-congruence) on X from $\preccurlyeq$.

A subset A of X is an ***ideal*** of X provided that $x \preccurlyeq y$ and $y \in A$ implies $x \in A$. For $a \in X$, we call $\{x \in X\colon\ x \preccurlyeq a\}$ the ***principal ideal generated*** by a and denote it by $\langle a \rangle$.

A bijection α of one pre-ordered set X onto another Y will be called an *isomorphism* provided that, for $a, b \in X$, $a \preccurlyeq b$ if and only if $a\alpha \preccurlyeq b\alpha$. In particular, if ν_X and ν_Y denote the respective mpo-congruences then $(a,b) \in \nu_X$ if and only if $(a\alpha, b\alpha) \in \nu_Y$.

Let $X(\preccurlyeq)$ be a pre-ordered set and ν the mpo-congruence from $\preccurlyeq$. Then X is a ***partially pre-ordered ϱ-set*** if and only if X/ν is a partially ordered ϱ-set with respect to the naturally induced order $\trianglelefteq$ from $\preccurlyeq$.

Let $X(\preccurlyeq)$ be a partially pre-ordered ϱ-set and σ an equivalence relation on X such that

- (O1) for any x in X, $\langle x \rangle$ is an ι-single subset with respect to σ,
- (O2) for x, y in X, if $(x,y) \in \sigma$ then $(x\nu, y\nu) \in \varrho$,
- (O3) for x, y, z in X, if $(x\nu)\varrho \wedge (y\nu)\varrho = (z\nu)\varrho$, $z_1\nu \trianglelefteq x\nu$ and $z_2\nu \trianglelefteq y\nu$ $(z_1\nu, z_2\nu \in (z\nu)\varrho)$, then for any $a \in \langle z_i \rangle$, there exists $b \in \langle z_j \rangle$ such that $(a,b) \in \sigma$, where $1 \le i,j \le 2$.

Then $(X(\preccurlyeq);\sigma)$ is called an *ω-set.*

Let $(X(\preccurlyeq);\sigma)$ be an ω-set and let $T_{(X(\preccurlyeq);\sigma)}$ denote the set of all isomorphisms from a principal ideal onto another one.

For any α, $\beta \in T_{(X(\preccurlyeq);\sigma)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(a,b) \in r(\alpha) \times d(\beta) : (a,b) \in \sigma\}.$$

Then $\theta_{\alpha,\beta} \in T_{(X(\preccurlyeq);\sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{(X(\preccurlyeq);\sigma)}\}$, and denote a multiplication $\circ$ and a unary operation $*$ on $T_{(X(\preccurlyeq);\sigma)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Clearly, $\alpha\circ\beta$ is an isomorphism from $\langle z_1\alpha^{-1}\rangle$ onto $\langle z_2\beta\rangle$. It is obvious that $T_{(X(\preccurlyeq);\sigma)}(\circ, *)$ is a regular $*$-semigroup. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$.

Theorem 3.1 *A regular $*$-semigroup $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ is a generalized inverse $*$-subsemigroup of $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$ whose set of projections is order isomorphic to X/ν.*

Remark 3.2 *In $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$, if $\preccurlyeq\, = \,\trianglelefteq$ and $\sigma = \varrho$ then $T_{(X(\trianglelefteq);\varrho)}(\mathcal{M}) = T_{X(\trianglelefteq)}(\mathcal{M})$.*

Let $(X(\preccurlyeq);\sigma)$ be an ω-set and let $Y = X/\nu$, where ν is the mpo-congruence from $\preccurlyeq$. For any element α in $T_{(X(\preccurlyeq);\sigma)}$, assume that $d(\alpha) = \langle a\rangle$. Then we can define a new mapping $\alpha' \in T_{Y(\trianglelefteq)}$ as follows:

$$d(\alpha') = \{x\nu : x \in d(\alpha)\},$$
$$(x\nu)\alpha' = (x\alpha)\nu.$$

Then $\alpha' \in T_{Y(\trianglelefteq)}$. Now, define a mapping $\xi : T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}) \to T_{Y(\trianglelefteq)}(\mathcal{M})$ by $\alpha\xi = \alpha'$. Then, it is easy to see that ξ is a $*$-homomorphism.

Proposition 3.3 *The mapping $\xi : \alpha \mapsto \alpha'$ of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ into $T_{Y(\trianglelefteq)}(\mathcal{M})$ is a $*$-homomorphism of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ onto a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{Y(\trianglelefteq)}(\mathcal{M})$ such that $\xi \circ \xi^{-1} = \mu$, where μ is the maximum idempotent separating $*$-congruence on $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$.*

Hereafter, we shall refer to ξ as the ***natural projection*** of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ to $T_{Y(\trianglelefteq)}(\mathcal{M})$.

3.2 Inflated representations

Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Define a relation $\preccurlyeq$ on S by:

$$a \preccurlyeq b \text{ if and only if } a^*a \leq b^*b,$$

for $a, b \in S$. Then clearly $\preccurlyeq$ is a pre-order on S for which the mpo-congruence from $\preccurlyeq$ is $\nu = \mathcal{L}$. Hence $S/\mathcal{L} = S/\nu$, under the naturally induced order $\trianglelefteq$ from $\preccurlyeq$, is just the set of $\mathcal{L}$-classes of S under the usual partial ordering of the $\mathcal{L}$-classes of a generalized inverse $*$-semigroup and so is order isomorphic to the partially ordered ϱ-set P of S. Hence S is a partially pre-ordered ϱ-set under $\preccurlyeq$. Then $\varrho = \mathcal{J}^E|_P$ and hence $(a\nu)\varrho(b\nu) \Longleftrightarrow a^*a\mathcal{J}^E b^*b$. Hereafter, for any $a \in S$, we think $a\nu = L_{a^*a}$ as a^*a.

For any $a \in S$, define a mapping $\rho_a : Sa^* \to Sa$ as follows:

$$d(\rho_a) = Sa^*(= Saa^*),$$
$$x\rho_a = xa.$$

Let $\rho : S \to \mathcal{GI}_{(S;\Omega)}(\mathcal{M})$ by $a\rho = \rho_a$, where the relation Ω defined by: for $x, y \in S$,

$$(x, y) \in \Omega \Longleftrightarrow x\rho_e = y \text{ for some } e \in E.$$

Since S is a regular $*$-semigroup, the representation ρ is faithful. Moreover, it follows from [6, Lemma 3.3] that it is a $*$-monomorphism.

Lemma 3.4 *The set $(S(\preccurlyeq); \Omega)$, defined above, is an ω-set.*

Again, we consider $\rho_a : Sa^* \to Sa$. By Lemma 3.4, $d(\rho_a) = \langle a^* \rangle$ and $r(\rho_a) = \langle a \rangle$. For $x, y \in d(\rho_a)$, $x^*x, y^*y \leq a^*a$. Now $x \preccurlyeq y$ if and only if $x^*x \leq y^*y$ while $xa \preccurlyeq ya$ if and only if $a^*x^*xa = (xa)^*(xa) \leq (ya)^*(ya) = a^*y^*ya$. But, since $x^*x, y^*y \leq a^*a$ it follows that $x^*x \leq y^*y$ if and only if $a^*x^*xa \leq a^*y^*ya$. Therefore $x \preccurlyeq y$ if and only if $xa \preccurlyeq ya$. Thus ρ_a is an isomorphism of $\langle a^* \rangle$ onto $\langle a \rangle$, and hence $S\rho \subseteq T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$.

Now, we have the following theorem.

Theorem 3.5 *Let S be a generalized inverse $*$-semigroup and define the relation $\preccurlyeq$ on S by $a \preccurlyeq b$ if and only if $a^*a \leq b^*b$. Then $\preccurlyeq$ is a pre-order on S with respect to which S is a partially pre-ordered ϱ-set, moreover $(S(\preccurlyeq); \Omega)$ is an ω-set. The faithful representation ρ of S embeds S as a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$.*

If ν is the mpo-congruence on S from $\preccurlyeq$, then $\nu = \mathcal{L}$ and S/ν is order isomorphic to the partially ordered ϱ-set P of S. Moreover, $\rho\xi = \tau$, where ξ is the natural projection and τ is the representation which is defined in Theorem 2.5.

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