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D-BOUNDED DISTANCE-REGULAR GRAPHS

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Let $\Gamma=(X, R)$ denote a distance-regular graph with distance function δ and diameter $D\geq 3$. A (vertex) subgraph $\Delta\subseteq X$ is said to be **weak-geodetically closed** whenever for all vertices $x, y\in\Delta$ and for all $z\in X$,

$$\delta(x, z)+\delta(z, y)\leq\delta(x, y)+1 \quad \longrightarrow \quad z\in\Delta.$$

It turns out that if Δ is weak-geodetically closed and regular then Δ is distance-regular. For each integer i $(0\leq i\leq D)$, Γ is said to be i-**bounded** whenever for all $x, y\in X$ at distance $\delta(x, y)\leq i$, x, y are contained in a common regular weak-geodetically closed subgraph of Γ of diameter $\delta(x, y)$. In [3], we assume $c_2>1$, $a_1\neq 0$, and characterize such Γ in terms of forbidden configurations.

Now assume Γ is D-bounded. Let $P(\Gamma)$ denote the poset whose elements are the weak-geodetically closed subgraphs of Γ, with partial order induced by reverse inclusion. Using $P(\Gamma)$, we obtain the following inequalities for the intersection numbers of Γ :

$$\frac{b_{D-i-1}-b_{D-i+1}}{b_{D-i-1}-b_{D-i}}\geq\frac{b_{D-i-2}-b_{D-i}}{b_{D-i-2}-b_{D-i-1}} \qquad (1\leq i\leq D-2).$$

We show equality is obtained in each of the above inequalities if and only if the intervals in $P(\Gamma)$ are modular. Moreover, we show this occurs if Γ has classical parameters and $D\geq 4$. This leads to our main result, which we now state.

Theorem A Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D\geq 4$. Suppose $b<-1$, and suppose the intersection numbers $a_1\neq 0$, $c_2>1$. Then

$$\beta=\alpha\frac{1+b^{D}}{1-b}.$$

(See [1] for the definition of distance-regular graphs with classical parameters.)

We use Theorem A to obtain the following results, which we believe are of independent interest.

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Theorem B Let Γ denote a distance-regular graph with diameter $D \geq 4$ and intersection number $c_2 > 1$. Then the following (i)-(ii) are equivalent.

(i) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with $b = -a_1 - 1$.

(ii) Γ is the dual polar graph ${}^2A_{2D-1}(-b)$.

Theorem C Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 4$. Assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Suppose Γ is a near polygon graph. Then Γ is a dual polar graph or a Hamming graph.

Theorem D Let Γ denote a distance-regular graph with diameter $D \geq 4$, and the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Then the following (i)-(ii) are equivalent.

(i) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with $b = -a_1 - 2$.

(ii) Γ is the Hermitian forms graph $Her_{-b}(D)$.

Using Hiroshi Suzuki's classification of D-bounded distance-regular graphs with $c_2 = 1$, $a_2 > a_1 > 1$[2], we prove the following result.

Theorem E There is no distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$, $D \geq 4$, $c_2 = 1$, and $a_2 > a_1 > 1$.

We would like to note that it is not necessary to assume the graph Γ is D-bounded in each of Theorem A-Theorem E.

REFERENCES

[1] A.E. Brouwer, A.M. Cohen, and A. Neumaier. Distance-Regular Graphs, Springer Verlag, New Tork, 1989.

[2] H. Suzuki. Strongly closed subgraphs of a distance-regular graph with geometric girth five. preprint.

[3] C. Weng. Weak-geodetically closed subgraphs in distance-regular graphs. preprint.