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Nonsymmetric Structure of Spin Models

東京医歯大　野村和正 (Kazumasa Nomura)

This is an *interim* report of a joint work with Francois Jaeger about nonsymmetric spin models and their link invariants. We mention here some of our results without their proofs.

1 Introduction

Spin models were introduced by Vaughan Jones [8] to obtain invariants of links and knots.

Definition. A *spin model* is a pair $S = (X, W)$ of a finite set X, $|X| = n > 0$, and a function

$$W : X \times X \longrightarrow \mathbf{C}^*$$

such that (for all a, b, $c \in X$)

$$\sum_{x \in X} \frac{W(a,x)}{W(b,x)} = 0 \qquad \text{if } a \neq b,$$

$$\frac{1}{\sqrt{n}} \sum_{x \in X} \frac{W(a,x)W(b,x)}{W(c,x)} = \frac{W(a,b)}{W(a,c)W(c,b)}.$$

The above two conditions are called the *type II* and *type III condition* respectively.

Remark. The function W can be viewed as an $n \times n$ matrix indexed by $X \times X$.

For each spin model $S = (X, W)$ and for each oriented link diagram L, there corresponds a complex number Z_L^S, and the correspondence

$$Z^S : L \longmapsto Z_L^S \in \mathbf{C}$$

gives a link invarinat, i.e.

$$L_1 \approx L_2 \Longrightarrow Z_{L_1}^S = Z_{L_2}^S,$$

where $L_1 \approx L_2$ means that two link diagrams L_1, L_2 represent isotopic links in 3-space.

Remark. The above definition of a spin model originally due to Vaughan Jones (for symmetric W). The definition was generalized to the general case (including nonsymmetric W) by Kawagoe-Munemasa-Watatani [9].

There exist many examples of nonsymmetric spin models. However, for each known nonsymmetric spin model S, we can find a symmetric spin model S' with $Z^S = Z^{S'}$. This leads to the following natural question.

Question. Does there exist a nonsymmetric spin model W whose link invariant does not come from any symmetric spin model?

Here we study nonsymmetric structure of spin models and give an answer to the above question.

2 Main Results

Theorem A. *For every spin model $S = (X, W)$, there exists a partition*

$$X = X_1 \cup \cdots \cup X_m$$

with $|X_1| = \cdots = |X_m|$ such that for all i, $j \in \{1, \ldots, m\}$ and for all $x \in X_i$, $y \in X_j$,

$$W(x, y) = \eta^{j-i} W(y, x)$$

holds, where $\eta = \exp(2\pi\sqrt{-1}/m)$.

Remark. From Theorem A, it is clear that

$$W(x, y) = W(y, x) \Longleftrightarrow x, y \in X_i \text{ for some } i$$

Hence $X_1, \ldots, X_m$ are the equivalence classes of the equivalence relation $\sim$ which is defined by $x \sim y$ iff $W(x, y) = W(y, x)$. In particular, m (The number of classes) is uniquely determined by S. We call m the (nonsymmetric) *index* of S. Obviously,

$$S \text{ has index } 1 \Longleftrightarrow W \text{ is symmetric.}$$

Theorem B. *If a spin model $S = (X, W)$ has odd index, then the link invariant of S agrees with the link invariant of some symmetric assciation scheme.*

We obtained new nonsymmetric spin models in the case of index 2:

Theorem C. *Let H be a Hadamard matrix of size $k \geq 4$, and let A be a square matrix of size k given by $A = (\alpha - \beta)I + \beta J$ with complex numbers α, β such that $\beta^2 + \beta^{-2} + \sqrt{k} = 0$, $\alpha = -\beta^{-3}$. Let W be a square matrix of size $n = 4k$ given by*

$$W = \begin{bmatrix} A & A & \eta H & -\eta H \\ A & A & -\eta H & \eta H \\ -\eta\,{}^tH & \eta\,{}^tH & A & A \\ \eta\,{}^tH & -\eta\,{}^tH & A & A \end{bmatrix}$$

where η is a primitive 8^{th}-root of unity. Then

(1) W satisfies type II and type III conditions, so that we have a non-symmetric spin model $S = (X, W)$ of index 2, where $X = \{1, ..., n\}$.

(2) The link invariant of the above spin model S does not agree with the link invariant of any symmetric spin model.

Thus the answer of the Question in the introduction is *YES.*

Remark. Jaeger and I are now trying to determine the link invariant of the above nonsymmetric spin model S.

3 Methods

In the proof of the results in the previous section, we essentially used the following results.

Theorem 1 (Jaeger-Matsumoto-Nomura [7]**).** *Let $S = (X, W)$, $|X| = n$, be a spin model. Then there exists a Bose-Mesner algebra $N(W)$ such that*

- $W \in N(W)$,
- *$N(W)$ has a duality $\Psi : N(W) \longrightarrow N(W)$ given by*

$$\Psi(A) = \frac{1}{\sqrt{n}\alpha}\, {}^tW^-({}^tW^+ \circ (W^-A)), \qquad A \in N(W),$$

where $\alpha = W(x,x)$ (independent of $x \in X$), $A \circ B$ denotes the Hadamard product: $(A \circ B)(x,y) = A(x,y)B(x,y)$, and $W^+ = W$, $W^-(x,y) = (W(y,x))^{-1}$.

Remark. See [2, 7] for definitions of Bose-Mesner algebras and their dualities.

Remark. The above theorem says that every spin model is obtained as a solution of modular invariance equations of some self-dual association scheme. This fact was proved by Jaeger [6] in the symmetric case (by topological methods). The algebra $N(W)$ was constructed for each symmetric type II matrix W by the author [12].

Remark. It is not so difficult to show that the matrix $E = \frac{1}{n}W^+ \circ W^-$ becomes an idempotent of rank 1 in $N(W)$. Hence $\Psi(E)$ is a permutation matrix contained in $N(W)$. This is one of the key obsevations of the proof of Theorem A, B, C.

Remark. Let E_0, E_1, ..., E_d be the primitive idempotents of the Bose-Mesner algebra $N(W)$. Then $\frac{1}{n}W^+ \circ W^- = E_s$ for some s. Put $\Psi(E_i) = A_i$, $i = 0, \ldots, d$, and let R_i be the relation on X with the adjacency matrix A_i $(i = 0, ..., d)$. Then the relations R_0, ..., R_d form an association scheme on X. In the proof of Proposision D below, we repeatedly used the following Lemma:

Lemma. *For every* x, $y \in X$,

$$(x, y) \in R_s \iff W(x, z) = W(z, y) \text{ for all } z \in X$$

In the proof of Theorem B, we need Bannai-Bannai's generalization of spin models: *4-weight spin model* defined in [1]. Theorem B is implied by Theorem 1 and the following result concerning "Gauge transformation" of 4-weight spin models.

Theorem 2 (Jaeger). *Let* $S = (X, W_1, W_2, W_3, W_4)$ *be a 4-weight spin model. Let* P *be a permutation matrix on* X *with* $PW_2 = W_2P$, *let* Δ *be an invertible diagonal matrix and let* λ *be a non-zero complex number. Then*

$$(X,\ \lambda\Delta W_1\Delta^{-1},\ \lambda^{-1}PW_2,\ \lambda^{-1}\Delta W_3\Delta^{-1},\ \lambda W_4\,{}^tP)$$

is a 4-weight spin model which gives the same link invariant as S.

Remark. A slightly weaker version of the above Theorem 2 was obtained independently by Deguchi [3].

Remark. In the case of odd index, we can find a permutation matrix $A_i \in N(W)$ with $A_i^2 = \Psi(E)$, where $E = \frac{1}{n} W^+ \circ W^-$. This is the reason why Theorem B holds in the case of odd index.

The spin model given in Theorem C is a nonsymmetric variation of the symmetric Hadamard model:

Theorem 3 (Nomura [12]). *Let H, A be matrices of size k defined in Theorem C. Let W be the square matrix of size $n = 4k$ given by*

$$W = \begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega\, {}^tH & -\omega\, {}^tH & A & A \\ -\omega\, {}^tH & \omega\, {}^tH & A & A \end{bmatrix},$$

where $\omega^4 = 1$. Put $X = \{1, \ldots, n\}$. Then $S = (X, W)$ is a symmetric spin model.

Remark. For a simpler proof of Theorem 3, see [11]. The link invariant Z^S of the above spin model S was determined by Jaeger [5, 6].

Theorem C is obtained from Theorem 3 and the following fact.

Proposition D.

(1) Let $S = (X, W)$ be a spin model with index 2. Then there is a partition

$$X = Y_1 \cup \cdots \cup Y_4$$

with $|Y_i| = (n/4)$, *and* W *splits into blocks, corresponding to* $Y_1, \ldots,$ Y_4, *as follows:*

$$W = \begin{bmatrix} A & A & B & -B \\ A & A & -B & B \\ -{}^tB & {}^tB & C & C \\ {}^tB & -{}^tB & C & C \end{bmatrix}.$$

Moreover A, B, C *satisfy type II condition, and* A, C *satisfy type III condition.*

(2) A matrix of the above form defines a spin model if and only if

$$W' = \begin{bmatrix} A & A & \eta B & -\eta B \\ A & A & -\eta B & \eta B \\ \eta\,{}^tB & -\eta\,{}^tB & C & C \\ -\eta\,{}^tB & \eta\,{}^tB & C & C \end{bmatrix},$$

defines a spin model, where η *is a primitive 8-root of unity.*

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