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ON SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE AND FIXED FINITELY MANY COEFFICIENTS

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ABSTRACT. The object of the present paper is to derive several interesting properties of the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$ consisting of regular and univalent meromorphic functions with positive and fixed finitely many coefficients. These include coefficient estimates, closure theorems, and radius of convexity for functions belonging to the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$.

KEY WORDS- Regular, univalent, meromorphic.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{ z : 0 < |z| < 1 \}$ with a simple pole at the origin with residue 1 there. And let Σ_p denote the subclass of Σ consisting of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \qquad (a_n \geq 0) \tag{1.2}$$

that are analytic and univalent in U^*. Recently Nunokawa, Aouf and Owa [2] investigated the class $\Sigma_p^*(\alpha,\beta,\mu)$ which is a subclass of Σ_p, defined as follows :

A function $f(z) \in \Sigma_p$ is in the class $\Sigma_p^*(\alpha,\beta,\mu)$ if it satisfies the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} + 1}{\mu \dfrac{zf'(z)}{f(z)} - 1 + (1+\mu)\alpha} \right| < \beta \qquad (z \in U^*) \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$, and μ $(0 \leq \mu \leq 1)$.

For the class $\Sigma_p^*(\alpha,\beta,\mu)$ Nunokawa, Aouf and Owa [2] showed the following lemma .

Lemma 1. Let the function f(z) be defined by (1.2). Then f(z) is in the class $\Sigma_p^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=1}^{\infty} \left\{ (n+1)+\beta\left[\mu n+(1+\mu)\alpha-1\right] \right\} a_n \leq (1+\mu)\beta(1-\alpha). \tag{1.4}$$

The result is sharp.

In view of Lemma 1, all functions belonging to the class $\Sigma_p^*(\alpha,\beta,\mu)$ satisfy the coefficient inequality

$$a_n \le \frac{(1+\mu)\beta(1-\alpha)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} \quad (n\ge 1). \tag{1.5}$$

Making use of (1.5), we now introduce the following class of functions:

Let $\Sigma_{p,c_k}^*(\alpha,\beta,\mu)$ denote the subclass of $\Sigma_p^*(\alpha,\beta,\mu)$ consisting of functions of the form

$$f(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]}z^i+\sum_{n=k+1}^{\infty}a_n z^n, \tag{1.6}$$

where

$$a_n\ge 0,\ 0\le c_i \le 1, \text{ and } 0\le \sum_{i=1}^{k} c_i \le 1 .$$

For k=1, the class $\Sigma_{p,c_1}^*(\alpha,\beta,\mu)=\Sigma_{p,c}^*(\alpha,\beta,\mu)$ was introduced by Aouf and Nunokawa [1].

In this paper,we obtain coefficient estimates and closure theorems for the class $\Sigma_{p,c_k}^*(\alpha,\beta,\mu)$. Further the radius of convexity is obtained for the class $\Sigma_{p,c_k}^*(\alpha,\beta,\mu)$. Techniques used are similar to those of Silverman and Silvia [4], Uralegaddi [5] and Owa and Srivastava [3] .

2. Coefficient Estimates

THEOREM 1. Let the function f(z) be defined by (1.6).Then f(z) is in the class $\Sigma_{p,c_k}^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=k+1}^{\infty}\left\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\right\}a_n\le (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i), \tag{2.1}$$

where

$$0 \le c_i \le 1 \text{ and } 0 \le \sum_{i=1}^{k} c_i \le 1.$$

The result (2.1) is sharp.

Proof. Putting

$$a_i = \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta\,[\mu i + (1+\mu)\alpha - 1]} \qquad (i=1,2,\ldots,k)\,, \tag{2.2}$$

in Lemma 1, we have

$$\sum_{i=1}^{k} (1+\mu)\beta(1-\alpha)c_i + \sum_{n=k+1}^{\infty} \left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} a_n \le (1+\mu)\beta(1-\alpha), \tag{2.3}$$

which clearly implies (2.1). Further, by taking the function $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta\,[\mu i + (1+\mu)\alpha - 1]} z^i$$

$$+ \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} z^n \tag{2.4}$$

for $n \ge k+1$, we can see that the result (2.1) is sharp.

Corollary 1. Let the function $f(z)$ defined by (1.6) be in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$. Then

$$a_n \le \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} \qquad (n \ge k+1)\quad . \tag{2.5}$$

The result (2.5) is sharp for the function $f(z)$ given by (2.4).

3. Closure Theorems

THEOREM 2. Let the functions

$$f_j(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]}z^i+\sum_{n=k+1}^{\infty}a_{n,j}z^n$$

$$(a_{n,j}\geq 0) \qquad (3.1)$$

be in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$ for every $j=1,2,...,m$. Then the function $F(z)$ defined by

$$F(z)=\sum_{j=1}^{m}d_jf_j(z) \qquad (d_j\geq 0) \qquad (3.2)$$

is also in the same class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$, where

$$\sum_{j=1}^{m}d_j=1\ . \qquad (3.3)$$

PROOF. Combining the definitions (3.1) and (3.2), we have

$$F(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]}z^i$$

$$+\sum_{n=k+1}^{\infty}\left(\sum_{j=1}^{m}d_j\,a_{n,j}\right)z^n, \qquad (3.4)$$

where we have also used the relationship (3.3). Since $f_j(z)\in\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$ for every $j=1,2,...,m$, Theorem 1 yields

$$\sum_{n=k+1}^{\infty}\left\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\right\}a_{n,j}\leq(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i), \qquad (3.5)$$

for every $j = 1,2, \ldots, m$. Thus we obtain

$$\sum_{n=k+1}^{\infty} \left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} \left[\sum_{j=1}^{m} d_j \, a_{n,j} \right]$$

$$= \sum_{j=1}^{m} d_j \left[\sum_{n=k+1}^{\infty} \left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} a_{n,j} \right]$$

$$\leq (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)$$

which (in view of Theorem 1) implies that $F(z) \in \Sigma^*_{p,c_k}(\alpha,\beta,\mu)$.

THEOREM 3. Let the functions $f_j(z)$ defined by (3.1) be in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$ for each $j=1,2,\ldots,m$, then the function $h(z)$ defined by

$$h(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta\,[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} b_n z^n \quad (b_n \geq 0) \tag{3.6}$$

is also in the same class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$, where

$$b_n = \frac{1}{m} \sum_{j=1}^{m} a_{n,j}\,. \tag{3.7}$$

PROOF. Since $f_j(z) \in \Sigma^*_{p,c_k}(\alpha,\beta,\mu)$, it follows from Theorem 1, that

$$\sum_{n=k+1}^{\infty} \left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} a_{n,j} \leq (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i). \tag{3.8}$$

Hence

$$\sum_{n=k+1}^{\infty}\left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} b_n$$

$$= \sum_{n=k+1}^{\infty}\left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} \left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j} \right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=k+1}^{\infty}\left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} a_{n,j} \right)$$

$$\leq (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i) \tag{3.9}$$

and the result follows.

Theorem 4. The class $\Sigma^{*}_{p,c_k}(\alpha,\beta,\mu)$ is closed under convex linear combination .

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (3.1) be in the class $\Sigma^{*}_{p,c_k}(\alpha,\beta,\mu)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z)= \lambda f_1(z) +(1-\lambda)f_2(z) \qquad (0\leq \lambda \leq 1) \tag{3.10}$$

is also in the class $\Sigma^{*}_{p,c_k}(\alpha,\beta,\mu)$.

Since

$$H(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta\ [\mu i + (1+\mu)\alpha -1]}\, z^i+\sum_{n=k+1}^{\infty}\{\lambda a_{n,1}+ (1-\lambda)a_{n,2}\}z^n, \tag{3.11}$$

we observe that

$$\sum_{n=k+1}^{\infty}\left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\}\left\{ \lambda a_{n,1}+ (1-\lambda)a_{n,2} \right\}$$

$$\leq (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i), \tag{3.12}$$

with the aid of Theorem 1. Hence $H(z) \in \Sigma^*_{p,c_k}(\alpha,\beta,\mu)$. This completes the proof of Theorem 4.

THEOREM 5. Let

$$f_k(z)=\frac{1}{z} + \sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta [\mu i + (1+\mu)\alpha -1]} z^i \tag{3.13}$$

and

$$f_n(z)=\frac{1}{z}+ \sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta [\mu i + (1+\mu)\alpha -1]} z^i$$

$$+ \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} z^n \qquad (n\geq k+1). \tag{3.14}$$

Then $f(z)$ is in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty}\lambda_n f_n(z) \tag{3.15}$$

where $\lambda_n\geq 0$ $(n\geq k)$ and

$$\sum_{n=k}^{\infty}\lambda_n = 1 . \tag{3.16}$$

PROOF. We suppose that $f(z)$ can be expressed in the form (3.15). Then it follows from (3.13), (3.14) and (3.16) that

$$f(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]}z^i$$

$$+\sum_{n=k+1}^{\infty}\frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i)\lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]}z^n . \qquad (3.17)$$

Note that

$$\sum_{n=k+1}^{\infty}\left\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\right\}\left\{\frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i)\lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]}\right\}$$

$$=(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i)\sum_{n=k+1}^{\infty}\lambda_n = (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i)(1-\lambda_k)$$

$$\le (1+\mu)\beta(1-\alpha)(1-\sum_{i=}^{k}c_i) , \qquad (3.18)$$

which implies that $f(z) \in \Sigma^*_{p,c_k}(\alpha,\beta,\mu)$.

For the converse, assume that the function $f(z)$ of the form (1.6) belongs to the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$.Since $f(z)$ satisfies (2.5) for $n\ge k+1$, we may set

$$\lambda_n = \frac{\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}}{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k}c_i)}a_n \le 1 \qquad (n\ge k+1) \qquad (3.19)$$

and

$$\lambda_k = 1-\sum_{n=k+1}^{\infty}\lambda_n . \qquad (3.20)$$

Hence $f(z)$ has the representation (3.15). This evidently completes the proof of Theorem 5.

4. Radius of Convexity

Theorem 6. Let the function $f(z)$ defined by (1.6) be in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$. Then $f(z)$ is meromorphically convex of order $\rho (0 \le \rho < 1)$ in the disc $0 < |z| < r = r(\alpha,\beta,\mu,c_i,\rho)$, where $r(\alpha,\beta,\mu,c_i,\rho)$ is the largest value for which

$$\sum_{i=1}^{k} \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1}$$

$$+ \frac{n(n+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} r^{n+1} \le 1-\rho , \qquad (4.1)$$

for $n \ge k+1$. The result is sharp for the function $f(z)$ given by (2.4).

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z)} + 2\right| \le 1-\rho \ (0 \le \rho < 1)$ for $0<|z|<r(\alpha,\beta,\mu,c_i,\rho)$. Note that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right|$$

$$\le \frac{\sum_{i=1}^{k} \frac{i(i+1)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} + \sum_{n=k+1}^{\infty} n(n+1)a_n r^{n+1}}{1-\sum_{i=1}^{k} \frac{i(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} - \sum_{n=k+1}^{\infty} na_n r^{n+1}}$$

$$\le 1-\rho \qquad (4.3)$$

for $0<|z|\le r$ if and only if

$$\sum_{i=1}^{k} \frac{(1+\mu)\beta(1-\alpha)i(i+2-\rho)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} + \sum_{n=k+1}^{\infty} n(n+2-\rho)a_n r^{n+1} \le 1-\rho. \qquad (4.4)$$

Since $f(z)$ is in the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$, from (2.5) we may take

$$a_n = \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)\ \lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} \qquad (n \geq k+1), \tag{4.5}$$

where $\lambda_n \geq 0$ $(n \geq k+1)$ and

$$\sum_{n=k+1}^{\infty} \lambda_n \leq 1. \tag{4.6}$$

For each fixed r, we choose the positive integer $n_o = n_o(r)$ for which $\dfrac{n(n+2-\rho)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]}\ r^{n+1}$ is maximal. Then it follows that

$$\sum_{n=k+1}^{\infty} n(n+2-\rho)a_n\ r^{n+1} \leq \frac{n_o(n_o+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n_o+1)+\beta[\mu n_o+(1+\mu)\alpha-1]}\ r^{n_o+1}. \tag{4.7}$$

Hence $f(z)$ is meromorphically convex of order ρ in $0<|z|<r(\alpha,\beta,\mu,c_i,\rho)$ provided that

$$\sum_{i=1}^{k} \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]}\ r^{i+1}$$

$$+ \frac{n_o(n_o+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n_o+1)+\beta[\mu n_o+(1+\mu)\alpha-1]}\ r^{n_o+1} \leq 1-\rho. \tag{4.8}$$

We find the value $r_o = r_o(\alpha,\beta,\mu,c_i,\rho)$ and the corresponding integer $n_o(r_o)$ so that

$$\sum_{i=1}^{k} \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r_o^{i+1} + \frac{n_o(n_o+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{(n_o+1)+\beta[\mu n_o+(1+\mu)\alpha-1]} r_o^{n_o+1}$$

$$= 1-\rho . \qquad (4.9)$$

Then this value r_o is the radius of meromorphically convex of order ρ for functions f(z) belonging to the class $\Sigma^*_{p,c_k}(\alpha,\beta,\mu)$.

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