On Certain Starlike Functions

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Abstract

Let $f(z)$ be analytic in $|z| < 1$, $f(0) = f'(0) - 1 = 0$ and suppose that

$$1 + \text{Re}(zf''(z)/f'(z)) < 3/2$$

in $|z| < 1$. Then, R. Singh and S. Singh [Colloquium Mathematicum, 47, 309-314 (1982)] proved that $f(z)$ is starlike in $|z| < 1$. The authors proved that if $f(z)$ is analytic in $|z| < 1$, $f(0) = f'(0) - 1 = 0$ and suppose that

$$1 + \text{Re}(zf''(z)/f'(z)) < 1 + (\alpha/2)$$

for $0 < \alpha \leq 1$, then we have

$$|\text{arg}(zf'(z)/f(z))| < (\pi\alpha)/2$$

in $|z| < 1$.

1 Introduction.

Let $A$ denote the class of functions $f(z)$ analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$.

A function $f(z) \in A$ is called starlike with respect to the origin if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{in} \quad U.$$ 

It is well known that every starlike function is univalent in $U$. Ozaki [2] proved that if $f(z) \in A$ and

$$1 + \text{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{3}{2} \quad \text{in} \quad U,$$

then $f(z)$ is univalent in $U$. R. Singh and S. Singh [4, Theorem 6] proved that if $f(z) \in A$ and satisfies the condition (1), then $f(z)$ is starlike in $U$.

In this paper, we need the following lemma.
Lemma 1. Let \( f(z) \in A \) and starlike with respect to the origin in \( U \).
Let \( C(r, \theta) = \{ f(te^{i\theta}) : 0 \leq t \leq r \} \) and let \( T(r, \theta) \) be the total variation of \( \arg f(te^{i\theta}) \) on \( C(r, \theta) \), so that
\[
T(r, \theta) = \int_0^r |\frac{\partial}{\partial t} \arg f(te^{i\theta})| dt.
\]
Then we have
\[
T(r, \theta) < \pi.
\]
We owe this lemma to Sheil-Small [5, Theorem 1].

2 Main result.

Main Theorem. Let \( f(z) \in A \) and
\[
1 + \text{Re} \frac{zf''(z)}{f'(z)} < 1 + \frac{\alpha}{2} \quad \text{in} \quad U,
\]
where \( 0 < \alpha \leq 1 \).
Then we have
\[
|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} \alpha \quad \text{in} \quad U
\]
or \( f(z) \) is starlike in \( U \).

Proof. Let us put
\[
\frac{2}{\alpha}(1 + \frac{\alpha}{2} - 1 - \frac{zf''(z)}{f'(z)}) = \frac{zg'(z)}{g(z)}
\]
where \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \).
From the assumption (2), we have that
\[
\text{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in} \quad U.
\]
This shows that \( g(z) \) is starlike and univalent in \( U \).
From (3) and by an easy calculation ( see e.g. [1] ), we have
\[
f'(z) = \left( \frac{g(z)}{z} \right)^{-\alpha/2}.
\]
Since \( g(z) \) is univalent in \( U \), we have that
\[
f'(z) \neq 0 \quad \text{in} \quad U.
\]
Therefore, we have

\[
\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)} dt \\
= \int_0^1 t^{\alpha/2} \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right)^{-\alpha/2} dt
\]

where \( z = re^{i\theta}, 0 \leq \theta < 2\pi \) and \( 0 < r < 1 \).

Since \( g(z) \) is starlike in \( U \), from Lemma 1, we have

\[
-\pi < \arg g(tre^{i\theta}) - \arg g(re^{i\theta}) < \pi
\]

for \( 0 < t \leq r \).

Putting

\[
s = t^{\alpha/2} \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right)^{-\alpha/2},
\]

then we have

\[
\arg s = -\frac{\alpha}{2} \arg \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right).
\]

From (5) and (6), \( s \) lies in the convex sector

\[
|\arg s| \leq \frac{\pi}{2}\alpha
\]

and the same is true of its integral mean of (4), (see e.g. [3, Lemma 1]).

Therefore we have

\[
|\arg \frac{f(z)}{zf'(z)}| < \frac{\pi}{2}\alpha \quad \text{in } U
\]

or

\[
|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2}\alpha \quad \text{in } U.
\]

This shows that

\[
\Re \frac{zf'(z)}{f(z)} > 0 \quad \text{in } U.
\]

This completes our proof and this is another proof of [4, Theorem 6].
References


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