数理解析研究所講究録
963巻 1996年 78-89

PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. For analytic functions $f(z)$ belonging to the class A_p, integral operators $I_n^a(f(z))$ are introduced. The object of the present paper is to derive some interesting properties of integral operators $I_n^a(f(z))$. Our results contain some previous results by M. Obradović ([4]), S. Owa, M. Obradović and M. Nunokawa ([6]), and by D. K. Thomas ([7]).

1. INTRODUCTION.

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. For functions $f(z)$ belonging to A_p, we define

$$I_0(f(z)) = \left(\frac{f(z)}{z^p}\right)^{\mu} \qquad (\mu > 0) \tag{1.2}$$

and

$$I_n^a(f(z)) = \frac{a+1}{z^{a+1}} \int_0^z t^a (I_{n-1}^a(f(t)))dt \qquad (a > -1), \tag{1.3}$$

where $n \in \mathbb{N}$ and $I_0^a(f(z)) = I_0(f(z))$. For $a = 0$, the operators $I_n^0(f(z))$ are introduced and studied by Owa [5].

Let A, B and λ be fixed real numbers such that $-1 \leqq B < A \leqq 1$ and $0 \leqq \lambda \leqq 1$. A function $f(z) \in A_p$ is said to be in the class $\mathcal{H}_p(A,B,\mu,\lambda)$ if it satisfies the condition

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1991 Mathematics Subject Classifications. Primary 30C45; Secondary 32A20.
Key Words and Phrases. Analytic function, hypergeometric series, subordination.

$$(1.4)\qquad (1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\mu} + \lambda\,\frac{f'(z)f(z)^{\mu-1}}{pz^{p\mu-1}} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}),$$

where the symbol " $\prec$ " stands for subordination. We note that $\mathcal{H}_1(A,B,\mu,\lambda)$ is a subclass of Bazilević functions (see [2]). Further, we get the following subclasses of analytic functions:

$$\mathcal{H}_p(1-2\alpha,-1,\mu,1) \equiv \mathcal{B}_p(\alpha,\mu)$$

$$= \left\{f(z) \in \mathcal{A}_p :\ \mathrm{Re}\left(\frac{f(z)}{z^p}\right)^{\mu} > \alpha,\ 0 \leqq \alpha < 1,\ z \in \mathcal{U}\right\}$$

and

$$\mathcal{H}_p(1-2\alpha,-1,\mu,0) \equiv \mathcal{C}_p(\alpha,\mu)$$

$$= \left\{f(z) \in \mathcal{A}_p :\ \mathrm{Re}\left(\frac{f'(z)f(z)^{\mu-1}}{pz^{p\mu-1}}\right) > \alpha,\ 0 \leqq \alpha < 1,\ z \in \mathcal{U}\right\}$$

In the present paper, we derive some interesting properties of integral operators $I_n^a(f(z))$ for functions $f(z)$ belonging to the class $\mathcal{H}_p(A,B,\mu,\lambda)$. Our results in this paper are the generalizations of the results by Obradović [4], Owa, Obradović and Nunokawa [6], and by Thomas [7].

2. Preliminaries and Main Results.

To establish our main results, we have to use the following lemmas.

Lemma 1 ([3]). If the function $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is analytic in $\mathcal{U}$, $h(z)$ is convex in $\mathcal{U}$ with $h(0) = 1$, and γ is a complex number such that $\mathrm{Re}(\gamma) > 0$, then the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

implies

$$p(z) \prec q(z) = \int_0^z t^{\gamma-1} h(t)\,dt \prec h(z) \qquad (z \in \mathcal{U})$$

and q(z) is the best dominant.

For complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the hypergeometric series

$$(2.1) \qquad {}_2F_1(a,b;c;z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots$$

represents an analytic function in $\mathbb{U}$. It is well known by [1] that

LEMMA 2. Let a, b, and c be real with $c \neq 0, -1, -2, \ldots$ and $c > b > 0$. Then

$$(2.2) \qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z),$$

$$(2.3) \qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a,c-b;c;z/(z-1)),$$

$$(2.4) \qquad {}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z},$$

and

$$(2.5) \qquad c(c-1)(z-1)\,{}_2F_1(a,b;c-1;z) + c\{c-1(2c-a-b-1)z\}\,{}_2F_1(a,b;c;z) + (c-a)(c-b)z\,{}_2F_1(a,b;c+1;z) = 0.$$

LEMMA 3. For any real number d $(d \neq 0)$, we have

$$(2.6) \qquad {}_2F_1(1,1;2;dz/(dz+1)) = \frac{(1+dz)\ln(1+dz)}{dz},$$

$$(2.7) \qquad {}_2F_1(1,1;3;dz/(dz+1)) = \frac{2(1+dz)}{dz}\left(1 - \frac{\ln(1+dz)}{dz}\right),$$

$$(2.8) \qquad {}_2F_1(1,1;4;dz/(dz+1)) = \frac{3(1+dz)}{2(dz)^3}\{2\ln(1+dz) - dz(2-dz)\},$$

and

$$(2.9) \qquad {}_2F_1(1,1;5;dz/(dz+1)) = \frac{2(1+dz)}{(dz)^3}\left(\frac{2(dz)^2 - 3dz + 6}{3} - \frac{2\ln(1+dz)}{dz}\right).$$

The proof of Lemma 3 follows from the identities (2.4) and (2.5).

Now, we derive

THEOREM. If $f(z) \in \mathcal{H}_p(A,B,\mu,\lambda)$ with $\lambda > 0$, then for $|z| = r < 1$

$$\text{(2.10)} \qquad \operatorname{Re}(I_n^a(f(z))) \geqq \rho_n(r) > \rho_n(1),$$

where $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, $I_0^a(f(z)) = I_0(f(z))$, and

$$0 < \rho_n(r) = 1 + (B-A)p\mu(a+1)^n \sum_{k=1}^{\infty} \frac{B^{k-1}r^k}{(p\mu+k\lambda)(k+a+1)^n} < 1.$$

The estimate in (2.10) is sharp.

PROOF. We shall prove this theorem by using the mathematical induction on n. Let us define the function p(z) by $p(z) = (f(z)/z^p)^\mu$ with $\mu > 0$. We choose a principal branch of $(f(z)/z^p)^\mu$ so that p(z) is analytic in $\mathbb{U}$ with $p(0) = 1$. Since f(z) is in the class $\mathcal{H}_p(A,B,\mu,\lambda)$, we see that

$$p(z) + \frac{\lambda}{p\mu} zp'(z) = (1-\lambda)\left(\frac{f(z)}{z^p}\right)^\mu + \lambda\frac{f'(z)f(z)^{\mu-1}}{pz^{p\mu-1}} \prec \frac{1 + Az}{1 + Bz}.$$

Thus by Lemma 1 with $\gamma = p\mu/\lambda$, we deduce that

$$\text{(2.11)} \qquad \left(\frac{f(z)}{z^p}\right)^\mu \prec \frac{p\mu}{\lambda} z^{-p\mu/\lambda} \int_0^z \frac{t^{(p\mu/\lambda)-1}(1+At)}{1+Bt} dt \equiv q(z)$$

$$= \frac{p\mu}{\lambda} \int_0^1 \frac{s^{(p\mu/\lambda)-1}(1+Asz)}{1+Bsz} ds$$

$$= \frac{p\mu}{\lambda} \int_0^1 s^{(p\mu/\lambda)-1}(1+Bsz)^{-1}ds + \frac{p\mu}{\lambda} Az\int_0^1 s^{p\mu/\lambda}(1+Bsz)^{-1}ds$$

By using (2.2) in (2.11), we get

$$\text{(2.12)} \qquad \left(\frac{f(z)}{z^p}\right)^\mu \prec q(z) = \begin{cases} {}_2F_1(1,p\mu/\lambda;1+p\mu/\lambda;-Bz) \\ \qquad + \dfrac{p\mu}{p\mu+\lambda} Az\,{}_2F_1(1,1+p\mu/\lambda;2+p\mu/\lambda;-Bz) & (B\neq 0) \\ 1 + \dfrac{p\mu}{p\mu+\lambda}Az & (B=0). \end{cases}$$

Now we show that

$$(2.13) \qquad \operatorname{Re}\{q(z)\} \geq q(-r) \qquad (|z| = r < 1).$$

Since $-1 \leq B < A \leq 1$, the function $(1+Az)/(1+Bz)$ is convex (univalent) in $\mathbb{U}$ and

$$\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br} > 0 \qquad (|z| = r < 1).$$

Setting

$$g(s,z) = \frac{1+Asz}{1+Bsz} \qquad (0 \leq s \leq 1,\ z \in \mathbb{U})$$

and

$$d\mu(s) = \frac{p\mu}{\lambda}\, s^{(p\mu/\lambda)-1} ds$$

which is a positive measure on $[0,1]$, we get from (2.11) that

$$(2.14) \qquad q(z) = \int_0^1 g(s,z)\,d\mu(s) \qquad (z \in \mathbb{U}).$$

Therefore, we have

$$\operatorname{Re}\{q(z)\} \geq \int_0^1 \operatorname{Re}\{g(s,z)\}\,d\mu(s) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\,d\mu(s)$$

which proves the inequality (2.13).

Now, using (2.13) in (2.12), we obtain

$$\operatorname{Re}\left(\frac{f(z)}{z^p}\right)^{\mu} \geq \rho_0(r) = \begin{cases} {}_2F_1(1,p\mu/\lambda;1+p\mu/\lambda;Br) \\ \qquad - \dfrac{p\mu}{p\mu+\lambda} Ar\,{}_2F_1(1,1+p\mu/\lambda;2+p\mu/\lambda;Br) & (B \neq 0) \\ 1 - \dfrac{p\mu}{p\mu+\lambda} Ar & (B = 0). \end{cases}$$

Simplifying the right hand side of the above estimate, we deduce that

$$(2.15) \qquad \operatorname{Re}\left(\frac{f(z)}{z^p}\right)^{\mu} \geq \rho_0(r) = \begin{cases} 1 + (B-A)p\mu \displaystyle\sum_{k=1}^{\infty} \frac{B^{k-1}r^k}{p\mu+k\lambda} & (B \neq 0) \\ 1 - \dfrac{p\mu}{p\mu+\lambda} Ar & (B = 0), \end{cases}$$

which implies that the result of the theorem is true for $n = 0$.

For $n = 1$, we have

$$\operatorname{Re}\{I_1^a(f(z))\} = \operatorname{Re}\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a\left(\frac{f(t)}{t^p}\right)^{\mu} dt\right\}$$

$$= \frac{a+1}{r^{a+1}}\int_0^r s^a \operatorname{Re}\left(\frac{f(se^{i\theta})}{se^{i\theta}}\right) ds$$

$$\geqq \frac{a+1}{r^{a+1}}\int_0^r s^a\left(1 + (B-A)p\mu \sum_{k=1}^{\infty} \frac{B^{k-1}s^k}{p\mu+k\lambda}\right) ds$$

$$= 1 + \frac{a+1}{r^{a+1}}(B-A)p\mu \int_0^r\left(\sum_{k=1}^{\infty} \frac{B^{k-1}s^{k+a}}{p\mu+k\lambda}\right) ds.$$

Since $|B| \leqq 1$, $s < 1$ and $p\mu+k\lambda \geqq p\mu+\lambda$ for all $k \geqq 1$, the series $\sum_{k=1}^{\infty} B^{k-1}s^{k+a}/(p\mu+k\lambda)$ is uniformly convergent in s, so that the series can be integrated term by term. Thus we have

$$\operatorname{Re}\{I_1^a(f(z))\} \geqq \rho_1(r) = 1 + (B-A)(a+1)p\mu \sum_{k=1}^{\infty} \frac{B^{k-1}r^k}{(p\mu+k\lambda)(k+a+1)}.$$

This shows that (2.10) holds true for $n = 1$.

Next, we assume that (2.10) holds true for $n = m$. Then, letting $t = se^{i\theta}$, we have

$$\operatorname{Re}\{I_{m+1}^a(f(z))\} = \operatorname{Re}\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a\{I_m^a(f(t))\}dt\right\}$$

$$= \frac{a+1}{r^{a+1}}\int_0^r s^a \operatorname{Re}\{I_m^a(f(t))\}dt$$

$$\geqq \frac{a+1}{r^{a+1}}\int_0^r s^a\left(1 + (B-A)(a+1)^m p\mu \sum_{k=1}^{\infty} \frac{B^{k-1}s^k}{(p\mu+k\lambda)(k+a+1)^m}\right) ds$$

$$= 1 + (B-A)(a+1)^{m+1}p\mu\frac{1}{r^{a+1}}\int_0^r\left(\sum_{k=1}^{\infty} \frac{B^{k-1}s^{k+a}}{(p\mu+k\lambda)(k+a+1)^m}\right) ds.$$

Noting that the integrand is uniformly convergent in s, we deduce that

$$\operatorname{Re}\{I_{m+1}^a(f(z))\} \geqq \rho_{m+1}(r) = 1 + (B-A)(a+1)^{m+1}p\mu \sum_{k=1}^{\infty} \frac{B^{k-1}r^k}{(p\mu+k\lambda)(k+a+1)^{m+1}}.$$

Therefore, we conclude that

$$\mathrm{Re}(I_n^a(f(z))) \geqq \rho_n(r)$$

for any integer $n \in \mathbb{N}_0$.

Finally, let us consider the function

$$\gamma_n(r) = 1 + (B-A)(a+1)^n p\mu \sum_{k=1}^{\infty} \frac{B^{k-1}r^k}{(p\mu+k\lambda)(k+a+1)^n} \qquad (0 < r < 1).$$

The series defined by $\gamma_n(r)$ is absolutely and uniformly convergent for each $n \in \mathbb{N}_0$ and $0 < r < 1$. By suitably rearranging the terms in $\gamma_n(r)$, it is easy to see that $0 < \gamma_n(r) < 1$. Further, since $\gamma_n(r) \leqq \gamma_{n-1}(r)$ and

$$r^{a+1}\gamma_n(r) = (a+1)\int_0^r s^a \gamma_{n-1}(s)ds \qquad (n \in \mathbb{N}),$$

we have that $\gamma_n'(r) < 0$ and $\gamma_n(r)$ decreases with r as $r \to 1^-$ for fixed n and increases to 1 as $n \to \infty$ for fixed r. This proves that $\gamma_n(r) > \gamma_n(1)$. The estimate in (2.10) is sharp for the function $f(z)$ given by

$$\left(\frac{f(z)}{z^p}\right)^{\mu} = \begin{cases} {}_2F_1(1, p\mu/\lambda; 1+p\mu/\lambda; -Bz) \\ \qquad + \dfrac{p\mu}{p\mu+\lambda} Az\,{}_2F_1(1, 1+p\mu/\lambda; 2+p\mu/\lambda; -Bz) & (B \neq 0) \\ 1 + \dfrac{p\mu}{p\mu+\lambda} Az & (B = 0), \end{cases}$$

where $\mu > 0$. This completes the proof of the theorem.

Setting $A = 1-2\alpha$ $(0 \leqq \alpha < 1)$, $B = -1$, and $\lambda = 1$ in Theorem, we have the following result which in turn leads to the result obtained by Owa [5] for $a = 0$.

COROLLARY 1. If $f(z) \in \mathcal{B}_p(\alpha,\mu)$, then

$$\mathrm{Re}\{I_n^a(f(z))\} \geqq \gamma_n(r) > \gamma_n(1) \qquad (n \in \mathbb{N}_0;\ |z| = r < 1),$$

where

$$0 < \gamma_n(r) = 1 - 2(1-\alpha)(a+1)^n p\mu \sum_{k=1}^{\infty} (-1)^{k-1} \frac{r^k}{(p\mu+k)(k+a+1)^n} < 1.$$

The result is sharp.

COROLLARY 2. If $f(z) \in A_1$ satisfies

$$f'(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in U)$$

with $-1 \leqq B < A \leqq 1$, then

$$(2.16) \qquad \frac{f(z)}{z} \prec q(z) = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)\dfrac{\ln(1+Bz)}{Bz} & (B \neq 0) \\ 1 + \dfrac{A}{2} z & (B = 0) \end{cases}$$

and $q(z)$ is the best dominant. Further,

$$(2.17) \qquad \operatorname{Re}\left(\frac{f(z)}{z}\right) > \begin{cases} \dfrac{A}{B} - \left(1-\dfrac{A}{B}\right)\dfrac{\ln(1-B)}{B} & (B \neq 0) \\ 1 - \dfrac{A}{2} & (B = 0). \end{cases}$$

The function $q(z)$ in (2.16) shows that the estimate in (2.17) is sharp.

PROOF. The first half of the corollary follows from (2.12) by taking $p = \mu = \lambda = 1$ and by using the identities (2.3), (2.4) and (2.7) respectively. The estimate (2.17) is obtained by letting $p = \mu = \lambda = 1$, $n = 0$ and $a = 0$ in (2.10). This completes the proof of the corollary.

In view of Corollary 2, we note that if $f(z) \in A_1$ satisfies the condition

$$f'(z) \prec \frac{1+A^{*}z}{1+Bz} \qquad (z \in U)$$

with $A^{*} = (B\ln(1-B))/(B+\ln(1-B))$, $B \neq 0$, then $\operatorname{Re}(f(z)/z) > 0$ $(z \in U)$. From this it follows that if $\operatorname{Re}(f'(z)) > (\ln 4 - 1)/(\ln 4 - 2)$, then $\operatorname{Re}(f(z)/z) > 0$ $(z \in U)$.

COROLLARY 3. Suppose that $f(z) \in A_1$ satisfies the condition

$$f'(z) + \lambda z f''(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U})$$

with $-1 \leqq B < A \leqq 1$ and $\lambda > 0$. Then

$$\text{(2.18)} \quad f'(z) \prec q(z) = \begin{cases} \dfrac{1}{1+Bz}\left[{}_2F_1(1,1;(\lambda+1)/\lambda;Bz/(Bz+1)) + \dfrac{Az}{\lambda+1}\, {}_2F_1(1,1;(2\lambda+1)/\lambda;Bz/(Bz+1))\right] & (B \neq 0) \\ 1 + \dfrac{Az}{\lambda+1} & (B = 0) \end{cases}$$

and this is the best dominant. Furthermore,

$$\text{(2.19)} \qquad \mathrm{Re}\{f'(z)\} > 1 + (B-A) \sum_{k=1}^{\infty} \frac{B^{k-1}}{(k\lambda+1)(k+1)^{n}} .$$

The function $q(z)$ in (2.18) shows that the estimate (2.19) is sharp.

PROOF. The result (2.18) is obtained by taking $zf'(z)$ instead of $f(z)$ and putting $p = \mu = 1$ in (2.12) followed by using the identity (2.3) The estimate (2.19) can be deduced by setting $p = \mu = 1$, $n = 0$ and $a = 0$ in (2.10).

Putting $\lambda = 1/2$ in Corollary 3, we have

EXAMPLE 1. If $f(z) \in \mathcal{A}_1$ satisfies

$$f'(z) + \frac{1}{2} z f''(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U}),$$

then

$$f'(z) \prec q(z) = \begin{cases} \dfrac{A}{B} - \dfrac{2}{B^2}\left(1 - \dfrac{A}{B}\right)\left(\dfrac{\ln(1+Bz) - Bz}{z^2}\right) & (B \neq 0) \\ 1 + \dfrac{2}{3} Az & (B = 0) \end{cases}$$

and this is the best dominant. Further,

$$\mathrm{Re}\{f'(z)\} > \begin{cases} \dfrac{A}{B} - \dfrac{2}{B^2}\left(1 - \dfrac{A}{B}\right)(\ln(1-B)+B) & (B \neq 0) \\ 1 - \dfrac{2}{3} A & (B = 0). \end{cases}$$

The result is sharp.

The above example for $A = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ was obtained by Owa, Obradović and Nunokawa [6]. We also observe that if $B \neq 0$ and

$$f'(z) + \frac{1}{2} zf''(z) \prec \frac{1 + A^{**}z}{1 + Bz} \qquad (z \in \mathbb{U})$$

with $A^{**} = \{2B(B+\ln(1-B))\}/\{2(B+\ln(1-B))+B^2\}$, then $\mathrm{Re}(f'(z)) > 0$ $(z \in \mathbb{U})$, and hence $f(z)$ is univalent in $\mathbb{U}$. This gives a new criterion for univalency.

In particular, if $B = -1$ then

$$\mathrm{Re}\{f'(z) + \frac{1}{2} zf''(z)\} > \frac{4\ln 2 - 3}{4\ln 2 - 2} = -0.2943\ldots$$

implies that $\mathrm{Re}(f'(z)) > 0$ $(z \in \mathbb{U})$.

Letting $\lambda = 1/3$ in Corollary 3, we have

EXAMPLE 2. If $f(z) \in \mathcal{A}_1$ satisfies

$$f'(z) + \frac{1}{3} zf''(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U}),$$

then

$$f'(z) \prec q(z) = \begin{cases} \dfrac{A}{B} + \dfrac{3}{(Bz)^3}\left(1 - \dfrac{A}{B}\right)\left\{\ln(1+Bz) - Bz + \dfrac{(Bz)^2}{2}\right\} & (B \neq 0) \\ 1 + \dfrac{3}{4} Az & (B = 0) \end{cases}$$

and this is the best dominant. Further,

$$\mathrm{Re}(f'(z)) > \begin{cases} \dfrac{A}{B} - \dfrac{3}{B^3}\left(1 - \dfrac{A}{B}\right)\left\{\ln(1-B)+B+\dfrac{B^2}{2}\right\} & (B \neq 0) \\ 1 - \dfrac{3}{4} A & (B = 0). \end{cases}$$

The result is sharp.

We observe that if $f(z) \in \mathcal{A}_1$ satisfies

$$\mathrm{Re}\{f'(z) + \frac{1}{3} zf''(z)\} > \frac{4 - 6\ln 2}{5 - 6\ln 2} = -0.1888\ldots \quad (z \in \mathbb{U}),$$

then $\mathrm{Re}(f'(z)) > 0$ $(z \in \mathbb{U})$.

Finally, taking $A = 1-2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $p = \mu = \lambda = 1$ and $n = 1$ in Theorem, we have

COROLLARY 4. If $f(z) \in A_1$ satisfies the condition

$$\mathrm{Re}(f'(z)) > \alpha \quad (z \in \mathbb{U})$$

for some α $(0 \leq \alpha < 1)$, then

$$\mathrm{Re}\left\{ \frac{a+1}{z^{a+1}} \int_0^z t^{a-1} f(t)\,dt \right\} > \rho \quad (z \in \mathbb{U}),$$

where

$$\rho = \rho(\alpha, a) = 1 - 2(1-\alpha)(a+1) \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)(k+a+1)} .$$

The result is sharp.

REMARK. (i) Putting $p = 1$, $\alpha = 0$ and $a = 0$ in Corollary 1, we get the corresponding result due to Thomas [7].

(ii) The results obtained by Owa [5, Corollary 3] can be deduced from Corollary 1 by taking $\mu = 1$, $\alpha = 0$ and $a = 0$.

(iii) Setting $A = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Corollary 2, we obtain the results contained in [4, Corollary 3].

(iv) Taking $A = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Example 1, we have the results due to Owa, Obradović and Nunokawa [6, Corollary 2].

ACKNOWLEDGMENTS. The present paper investigation of the first author was supported, in part, by the Japanese Ministry of Education, Science and Culture under a grant-in-aid for general scientific research.

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