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On certain subclasses of analytic functions involving a linear operator

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Abstract

A certain linear operator defined by a Hadamard product or convolution for functions which are analytic in the open unit disk is introduced according to Carlson and Shaffer. The purpose of the present paper is to give some properties of this linear operator. Our results contain the earlier theorems in the univalent and multivalent function theory.

1 Introduction

Let $\mathbf{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N = \{1, 2, 3, \cdots\}) \tag{1. 1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$, and $\mathbf{A}_1 = \mathbf{A}$.

For

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \ ,$$

we define the Hadamard product (or convolution) by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \ .$$

Let

$$\phi_p(a, c, ; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (c \neq 0, -1, -2, \cdots ; \ \ z \in U) \ , \tag{1. 2}$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1 & (\text{if } n = 0) \\ x(x+1)(x+2)\cdots(x+n-1) & (\text{if } n \in N) \end{cases} \ .$$

We note that

$$\phi_p(a, c; z) = z^p \cdot {}_2F_1(1, a; c; z) \ ,$$

where

$${}_2F_1(1, a; c; z) = \sum_{n=0}^{\infty} \frac{(1)_n (a)_n}{(c)_n} \frac{z^n}{n!} \ .$$

Remark 1.

$$\begin{aligned} \phi_p(a, 1; z) &= \frac{z^p}{(1-z)^a} \\ \phi_p(a, 1; z) &= \frac{z}{(1-z)^2} \ \ (\text{koebe function}) \ . \end{aligned}$$

Corresponding to the function $\phi_p(a, c; z)$, we define a linear operator $L_p(a, c)$ on $\mathbf{A}_p$ by the convolution

$$L_p(a, c) f(z) = \phi_p(a, c; z) * f(z) \tag{1. 3}$$

for $f(z) \in \mathbf{A}_p$.

Remark 2. If $c > a > 0$, $L_p(a,c)$ has integral representation

$$L_p(a,c)f(z) = \int_0^1 u^{-p} f(uz) d\mu(u) \ ,$$

where μ satisfies

$$d\mu(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{B(a,c-a)} du \quad \text{and} \quad \int_0^1 d\mu(u) = 1 \ .$$

Clearly, $L_p(a,a)$ is the unit operator and

$$L_p(a,c) = L_p(a,b)L_p(b,c) = L_p(b,c)L_p(a,b) \quad (b,c \neq 0,-1,-2,\cdots) \ .$$

Moreover, if $a \neq 0,-1,-2,\cdots$, then $L_p(a,c)$ has an inverse $L_p(c,a)$.

The operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex and prestarlike hypergeometric functions.

In recent years Srivatava and Owa [15] have given some properties of $L_1(a,c)$ concerning with univalent functions in U.

Remark 3. For $f(z) \in \mathbf{A}_p$,

$$L_p(\nu+p,1)f(z) = \frac{z^p}{(1-z)^{\nu+p}} * f(z) = D^{\nu+p-1} f(z) \quad , \tag{1. 4}$$

where $\nu(> p)$ is any real number . In the case of $p = 1$ and $\nu \in N$, $D^\nu f(z)$ is the Ruscheweyh derivative [8] .

$$L_p(\nu+p,\nu+p+1)f(z) = \frac{\nu+p}{z^\nu} \int_0^z t^{\nu-1} f(t) dt = J_{\nu,p}(f(z)) \ , \tag{1. 5}$$

where $\nu + p > 0$. The operator $J_{\nu,1}(\nu \in N)$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some results for the operator $J_{\nu,p}$ were showed by Saitoh [10] and Saitoh et al [14].

A function $f(z)$ belonging to the class $\mathbf{A}_1$ is said to be starlike if and only if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in U) \ . \tag{1. 6}$$

A function $f(z)$ belonging to the class $\mathbf{A}_1$ is convex if and only if it satisfies

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U) \ . \tag{1. 7}$$

The subclass of univalent functions consisting of the starlike functions is denoted by S^*, and K denotes the subclass of convex functions.

A function $f(z) \in \mathbf{A}_p$ is said to be p-valently starlike if and only if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in U) \ . \tag{1. 8}$$

A function $f(z) \in \mathbf{A}_p$ is convex if and only if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U) \ . \tag{1. 9}$$

2 Some properties of univalent and multivalent functions involving the operator $L_p(a, c)$

In order to prove our results we need the following lemmas.

Lemma 1. (Saitoh [11]) *If* $f(z) \in \mathbf{A}_p$, *then*

$$z(L_p(a,c)f(z))' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z) \tag{2. 1}$$

where $c \neq 0, -1, -2, \cdots$.
Proof. Note that

$$L_p(a,c)f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} \qquad (a_p = 1)$$

and

$$L_p(a+1,c)f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} \qquad (a_p = 1) \ .$$

These give that

$$\begin{aligned} &aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z) \\ &= \sum_{n=0}^{\infty} (a+n)\frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} - \sum_{n=0}^{\infty} (a-p)\frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} \\ &= \sum_{n=0}^{\infty} (n+p)\frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} \\ &= z(L_p(a,c)f(z))' \ . \end{aligned}$$

Lemma 2. (Saitoh and Nunokawa [12]) *Let* $g(z)$ *be analytic and satisfies* $\operatorname{Re}\{g(z)\} > 0$ *in* U.
Then we have

$$|zg'(z)| \leq \frac{2|z|\operatorname{Re}\{g(z)\}}{1-|z|^2} \qquad (z \in U) \ . \tag{2. 2}$$

Lemma 3. *If* $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ *is analytic in* U *and* $\mathrm{Re}\{g(z)\} > 0$. *Then*

$$\mathrm{Re}\{g(z)\} \geq \frac{1-|z|}{1+|z|} \quad . \tag{2. 3}$$

Lemma 4. (MacGregor [7]) *Suppose that* $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is analytic and satisfies* $\mathrm{Re}\{g(z)\} > 0$ *in* U. *Then*

$$\left|\frac{h'(z)}{h(z)}\right| \leq \frac{2}{1-|z|^2} \qquad (z \in U) \ . \tag{2. 4}$$

Applying lemmas, we prove

Theorem 1. *Let* $f(z) \in \mathbf{A}_p$ *and* $a > 0$. *If*

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{z^p}\right\} > 0 \qquad (z \in U) \ ,$$

then

$$\begin{aligned} \mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} &\geq \frac{a-2r-ar^2}{a(1+r)^2} \\ \text{for } |z| &= r < \frac{\sqrt{1+a^2}-1}{a} \ . \end{aligned} \tag{2. 5}$$

Proof. We put $g(z) = \dfrac{L_p(a,c)f(z)}{z^p}$.
Then $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic and satisfies $\mathrm{Re}\{g(z)\} > 0$ in U.
Differentiating of $z^p g(z) = L_p(a,c)f(z)$ and using Lemma 1, we have

$$\frac{L_p(a+1,c)f(z)}{z^p} = g(z) + \frac{1}{a} z g'(z) \ .$$

Therefore, using Lemma 2 and Lemma 3

$$\begin{aligned} \mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} &\geq \mathrm{Re}\{g(z)\} - \frac{1}{a}|zg'(z)| \\ &\geq \mathrm{Re}\{g(z)\} - \frac{2|z|\mathrm{Re}\{g(z)\}}{a(1-|z|^2)} \\ &= \frac{\mathrm{Re}\{g(z)\}(a-2|z|-a|z|^2)}{a(1-|z|^2)} \\ &\geq \frac{1-|z|}{1+|z|}\frac{a-2|z|-a|z|^2}{a(1-|z|^2)} \\ &= \frac{a-2r-ar^2}{a(1+r)^2} \\ \text{for } |z| &= r < \frac{\sqrt{1+a^2}-1}{a} \ . \end{aligned}$$

This completes the proof of Thm. 1.

Putting $a = c = p$ in Thm. 1, we have

Corollary 1. (Saitoh and Nunokawa [12]) *Let* $f(z) \in \mathbf{A}_p$. *If*

$$Re\left\{\frac{f(z)}{z^p}\right\} > 0 \qquad (z \in U) ,$$

then we have

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} \geq \frac{p-2r-pr^2}{(1+r)^2} \quad \textit{for} \quad |z| = r < \frac{\sqrt{1+p^2}-1}{p} . \tag{2. 6}$$

Corollary 2. (Yamaguchi [16]) *Let* $f(z) = z + a_2z^2 + \cdots$ *be analytic in* U . *If* $Re\left\{\frac{f(z)}{z}\right\} > 0$, *then we have*

$$Re\{f'(z)\} > 0 \quad \textit{for} \quad |z| = r < \sqrt{2}-1 .$$

Next, we prove

Theorem 2. *Let* $f(z) \in \mathbf{A}_p$ *and* $a > 0$. *If*

$$Re\left\{\frac{L_p(a,c)f(z)}{z^p}\right\} > 0 \qquad (z \in U) ,$$

then

$$\left|\frac{aL_p(a+1,c)f(z)}{L_p(a,c)f(z)} - a\right| \leq \frac{2|z|}{1-|z|^2} \qquad (z \in U) . \tag{2. 7}$$

Proof. We put $h(z) = \dfrac{L_p(a,c)f(z)}{z^p}$. Then $h(z) = 1 + c_1z + c_2z^2 + \cdots$ is analytic and satisfies $\mathrm{Re}\{h(z)\} > 0$ in U. Logarithmically differentiating of $h(z)$ and using Lemma 1, we can see

$$\frac{aL_p(a+1,c)f(z)}{L_p(a,c)f(z)} - a = \frac{zh'(z)}{h(z)} .$$

From Lemma 4, we have

$$\left|\frac{aL_p(a+1,c)f(z)}{L_p(a,c)f(z)} - a\right| = \left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2|z|}{1-|z|^2} .$$

This completes the proof of Thm. 2 .

Letting $a = c = p$ in Thm. 2, we have the following interesting Corollary.

Corollary 3. *Let* $f(z) \in A_p$. *If*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\} > 0 \qquad (z \in U) ,$$

then $f(z)$ *is p-valently starlike in* $|z| < \dfrac{\sqrt{1+p^2}-1}{p}$.

Proof. From Theorem 2,

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{2|z|}{1-|z|^2} < p \ \text{ for } \ |z| < \frac{\sqrt{1+p^2}-1}{p}$$

$$\text{i.e.,} \quad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \text{ in } \ |z| < \frac{\sqrt{1+p^2}-1}{p} .$$

Further, taking the function $f(z)$ given by

$$f(z) = \frac{z^p(1-z)}{1+z} ,$$

we see that the result is sharp.

Corollary 4. (MacGregor [6]) *Let* $f(z) = z + a_2z^2 + \cdots$ *be analytic and satisfies* $\mathrm{Re}\left\{\dfrac{f(z)}{z}\right\} > 0$ *in* U. *Then* $f(z)$ *is starlike in* $|z| < \sqrt{2}-1$.

Using the same technique of Thm. 2, we prove

Theorem 3. *Let* $f(z) \in A_p, a > 0$. *If*

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{z^p}\right\} > 0 \qquad (z \in U) ,$$

then

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right\} \geq \frac{a-2|z|-a|z|^2}{a(1-|z|^2)} \qquad (z \in U) . \tag{2. 8}$$

Proof. We put $h(z) = \dfrac{L_p(a,c)f(z)}{z^p}$.

Then $h(z) = 1 + c_1z + c_2z^2 + \cdots$ is analytic and satisfies $\mathrm{Re}\{h(z)\} > 0$ in U. Differentiating of $h(z)$ and using Lemma 1, we have

$$\frac{L_p(a+1,c)f(z)}{L(a,c)f(z)} = 1 + \frac{1}{a}\frac{zh'(z)}{h(z)} .$$

Therefore, from Lemma 4 we can see that

$$\begin{aligned}\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right\} &\geq 1 - \frac{1}{a}\left|\frac{zh'(z)}{h(z)}\right| \\ &\geq 1 - \frac{1}{a}\frac{2|z|}{1-|z|^2} \\ &= \frac{a-2|z|-a|z|^2}{a(1-|z|^2)}\end{aligned}$$

This completes the proof of Thm. 3.

Making $a = c = p$ in Thm. 3, we can show Cor. 3.

Next, we prove

Theorem 4. *Let* $f(z), g(z) \in \mathbf{A}_p, a > 0$ *and satisfies*

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}\right\} > 0 \qquad (z \in U) \ .$$

If

$$\operatorname{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > 0 \qquad (z \in U) \ ,$$

then

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right\} > 0 \ \ \textit{for} \ \ |z| < \frac{a+1-\sqrt{2a+1}}{a} \ . \tag{2. 9}$$

Proof. We put $p(z) = \dfrac{L_p(a,c)f(z)}{L_p(a,c)g(z)}$. Then $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic and satisfies $\operatorname{Re}\{p(z)\} > 0$ in U . Differentiating of

$$L_p(a,c)f(z) = p(z) \cdot L_p(a,c)g(z) \ ,$$

we have

$$z(L_p(a,c)f(z))' = zp'(z)L_p(a,c)g(z) + p(z)z(L_p(a,c)g(z))'$$

From Lemma 1, we can see

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} = \frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)} + \frac{1}{a}\frac{zp'(z)}{p(z)} \ .$$

Now, we put $q(z) = \dfrac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}$.
Then $q(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic and satisfies $\operatorname{Re}\{q(z)\} > 0$ in U . Therefore, using Lemma 3 and Lemma 4, we have

$$\begin{aligned}
\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right\} &\geq \operatorname{Re}\{q(z)\} - \frac{1}{a}\left|\frac{zp'(z)}{p(z)}\right| \\
&\geq \frac{1-|z|}{1+|z|} - \frac{1}{a}\frac{2|z|}{1-|z|^2} \\
&= \frac{a|z|^2 - 2(a+1)|z| + a}{a(1-|z|^2)} \\
&> 0 \ \text{in} \ |z| < \frac{a+1-\sqrt{2a+1}}{a} \ .
\end{aligned}$$

This completes the proof of Thm. 4.

Putting $a = c = p$ in Thm. 4, we have

Corollary 5. *Let* $f(z), g(z) \in \mathbf{A}_p$ *and satisfies*

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \qquad (z \in U) .$$

If

$$\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \qquad (z \in U) ,$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \ in \ \ |z| < \frac{p+1-\sqrt{2p+1}}{p} . \tag{2. 10}$$

Letting $p = 1$ in Cor. 5, we can see

Corollary 6. (MacGregor [7]) *Let* $f(z), g(z) \in \mathbf{A}$ *and satisfies* $\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \quad (z \in U)$. *If* $\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad (z \in U)$, *then* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ *in* $|z| < 2-\sqrt{3}$.

Now, we prove the next theorem .

Theorem 5. *Let* $f(z) \in \mathbf{A}_p$ *and* $a > 0$. *If*

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} > 0 \qquad (z \in U) ,$$

then

$$\operatorname{Re}\left\{\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}\right\} \geq \frac{(a+1)-2|z|-(a+1)|z|^2}{(a+1)(1-|z|^2)} . \tag{2. 11}$$

Proof. We put $g(z) = \dfrac{L_p(a+1,c)f(z)}{z^p}$. Then $g(z) = 1 + a_1 z + a_2 z^2 + \cdots$ is analytic and satisfies $\operatorname{Re}\{g(z)\} > 0$ in U. Differentiating $z^p \cdot p(z) = L_p(a+1,c)f(z)$ and using Lemma 1, we have

$$p + \frac{zg'(z)}{g(z)} = (a+1)\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} - (a+1-p) ,$$

that is,

$$\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} = 1 + \frac{1}{a+1} \cdot \frac{zg'(z)}{g(z)} .$$

Therefore, from Lemma 4

$$\begin{aligned}\mathrm{Re}\left\{\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}\right\} &\geq 1-\frac{1}{a+1}\left|\frac{zg'(z)}{g(z)}\right| \\ &\geq 1-\frac{1}{a+1}\cdot\frac{2|z|}{1-|z|^2} \\ &= \frac{(a+1)-2|z|-(a+1)|z|^2}{(a+1)(1-|z|^2)} \ .\end{aligned}$$

This completes the proof of Thm. 5,

Letting $a=c=p$ in Thm. 5, we have

Corollary 7. *Let* $f(z)\in \mathbf{A}_p$. *If*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^p-1}\right\}>0 \qquad (z\in U) \ ,$$

then

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>0 \ \ in \ \ |z|<\frac{\sqrt{1+p^2}-1}{p} \ . \tag{2. 12}$$

Proof. By easy calculation, we have

$$\mathrm{Re}\left\{\frac{L_p(p+2,p)f(z)}{L_p(p+1,p)f(z)}\right\}=\mathrm{Re}\left\{\frac{2f'(z)+f''(z)}{(p+1)f'(z)}\right\}=\frac{1}{p+1}\left(2+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right) \ .$$

Therefore, from Thm. 5, we have

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\geq\frac{p-2|z|-p|z|^2}{1-|z|^2} \qquad (z\in U) \ .$$

Putting $p=1$ in Cor. 7, we have

Corollary 8. (MacGregor [6]) *Let* $f(z)=z+a_2z^2+\cdots$ *be analytic in* U. *If*

$$\mathrm{Re}\{f'(z)\}>0 \qquad (z\in U) \ ,$$

then

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>0 \ \ in \ \ |z|<\sqrt{2}-1 \ .$$

In order to prove next theorem, we need the following lemma due to Ponnusamy .

Lemma 5. *If* $p(z)$ *is analytic in* U *with* $p(0)=1$, *and if* λ *is a complex number satisfying* $\mathrm{Re}\lambda\geq 0$ $(\lambda\neq 0)$, *then*

$$\mathrm{Re}\{p(z)+\lambda zp'(z)\}>\alpha \quad (0\geq\alpha<1) \tag{2. 13}$$

implies

$$\operatorname{Re}\{p(z)\} > \alpha + (1-\alpha)(2\gamma - 1) \ , \tag{2. 14}$$

where γ *is given by*

$$\gamma = \gamma(\operatorname{Re}\lambda) = \int_0^1 \frac{dt}{1 + t^{\operatorname{Re}\lambda}} \ . \tag{2. 15}$$

is an increasing function of $\operatorname{Re}\lambda$ *and* $\frac{1}{2} \leq \gamma < 1$. *The estimate is sharp in the sense that the bound cannot be improved.*

Applying Lemma 5, we prove

Theorem 6. *Let* λ *be a complex number satisfying* $\operatorname{Re}\lambda > 0$. *Let* $f(z) \in \mathbf{A}_p$ *satisfies the condition*

$$\operatorname{Re}\left\{(1-\lambda)\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu} + \lambda \cdot \frac{L_p(a+1,c)f(z)}{z^p}\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu-1}\right\} > \alpha \qquad (z \in U) \ , \tag{2. 16}$$

for some α $(0 \leq \alpha < 1)$ *and* $\mu > 0$. *Then*

$$\operatorname{Re}\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu} > \alpha + (1-\alpha)(2\gamma - 1) \quad (z \in U) \ , \tag{2. 17}$$

where $\gamma = {}_2F_1\left(1, \frac{\mu a}{\operatorname{Re}\lambda}; 1 + \frac{\mu a}{\operatorname{Re}\lambda}; -1\right)$. *The estimate is sharp.*

Proof. We put

$p(z) = \left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu}$, then $p(z)$ is analytic in U and $p(0) = 1$. Using Lemma 1, we have

$$\begin{aligned} p(z) &+ \frac{\lambda}{a\mu} z p'(z) \\ &= (1-\lambda)\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu} + \lambda \frac{L_p(a+1,c)f(z)}{z^p}\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{\mu-1} \end{aligned}$$

From assumption,

$$\operatorname{Re}\left\{p(z) + \frac{\lambda}{a\mu} z p'(z)\right\} > \alpha \quad (0 \leq \alpha < 1, \ \mu > 0) \ .$$

Therefore, according to Lemma 5, we have

$$\operatorname{Re}\{p(z)\} > \alpha + (1-\alpha)(2\gamma - 1) \ ,$$

where

$$\gamma = \gamma(\mathrm{Re}\frac{\lambda}{a\mu}) = \int_0^1 \frac{dt}{1+t^{\mathrm{Re}(\lambda/a\mu)}} \ .$$

We set $\mathrm{Re}\lambda = \lambda_1$, then we have

$$\begin{aligned} \gamma &= \int_0^1 \frac{dt}{1+t^{\lambda_1/a\mu}} = \frac{a\mu}{\lambda_1}\int_0^1 \mu^{a\mu/\lambda_1-1}(1+\mu)^{-1}du \\ &= {}_2F_1(1,\frac{a\mu}{\lambda_1};1+\frac{a\mu}{\lambda_1};-1) \ . \end{aligned}$$

Putting $\mu = 1, \lambda = 1$ in Thm. 6, we have

Corollary 9. *Let $f(z) \in \mathbf{A}_p$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} > \alpha \qquad (z \in U) \ , \tag{2. 18}$$

for some α $(0 \le \alpha < 1)$. Then

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{z^p}\right\} > \alpha + (1-\alpha)(2\gamma - 1) \qquad (z \in U) \ , \tag{2. 19}$$

where $\gamma = {}_2F_1(1,a;a+1;-1)$. The estimate is sharp.

Taking $a = c = 1$ and $p = 1$ in Cor. 9, we can have the following well-known result.

Corollary 10. *Let $f(z) = z + a_2z^2 + \cdots$ be analytic in U satisfying*

$$\mathrm{Re}\{f'(z)\} > \alpha \qquad (0 \le \alpha < 1) \ . \tag{2. 20}$$

Then we have

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \alpha + (1-\alpha)(2\gamma - 1) \ , \tag{2. 21}$$

where

$$\gamma = {}_2F_1(1,1;2;-1) = ln2 \ .$$

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