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Distortion Theorem for Fractional Integral Operator

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Abstract

We define the generalized subclasses of the class consisting of analytic functions with negative coefficients. We obtain the distortion theorem on functions in the generalized subclasses for fractional integral operator involving the generalized hypergeometric function.

1 Introduction and Definitions

Let $A(n,p)$ denote the class of functions of the form

$$f(z)=z-\sum_{k=n+p}^{\infty}a_kz^k \quad (a_k\geq 0;\ n,p\in N) \tag{1}$$

that are analytic in the unit disk $U=\{z:|z|<1\}$. Let $A(n,p,\{B_k\})$ denote the subclass of $A(n,p)$ consisting of functions which satisfy the following inequality:

$$\sum_{k=n+p}^{\infty}B_ka_n\leq 1 \quad (B_k>0;\ n,p\in N). \tag{2}$$

The subclasses $A(n,p;\{B_k\})$ is called the generalized subclasses of the class consisting of analytic functions with negative coefficients. The case of p=1 and the case of arbitraly positive integer p were considered by Sekine [4], and Owa and Obradvic[3], respectively.

In [4], we expressed various known subclasses of the class consisting of the functions with negative coefficients in terms of the generalized subclasses and obtained inclusion relation of these classes. We gave distortion theorems on the derivatives of integer order of functions belonging to the generalized classes. Further using the fractional integral,

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fractional derivative by Owa[2] and the fractional integral operator by Srivastava, Saigo and Owa[6], we extend the distortion theorems on the derivative of arbitrary order of functions in the generalized subclasses.

Let $\alpha_j (j = 1, \cdots, p)$ and $\beta_j (j = 1, \cdots, q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \cdots \quad (j = 1, \cdots, q).$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$\begin{aligned} {}_pF_q &\equiv {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) \\ &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q+1), \end{aligned} \tag{3}$$

where $(\lambda)_n$ is the pohhammer symbol defined by

$$(\lambda)_n \stackrel{\text{def}}{=} \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in N = \{1, 2, 3, \cdots\}, \\ 1, & n = 0. \end{cases} \tag{4}$$

Many essentially equivalent definitions of fractional calculus have been given. We state the following definitions due to Owa[2] which have been used rather frequentry in the theory of analytic function:

Definition 1. 1 (Owa[2]) *The fractional integral of order λ is defined by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \tag{5}$$

where $\lambda > 0$, f(z) is an analytic function in a simply connected region of the z-plane containing the origin and the many-valuedness of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 1. 2 (Owa[2]) *The fractional derivative of order λ is defined by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{6}$$

where $0 \leq \lambda < 1$, f(z) is an analytic function in a simply connected region of the z-plane containing the origin and the many-valuedness of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 1.3 (Owa[2]) *Under the hypotheses of Definitio 1.2, the fractional derivative of order $(n+\lambda)$ is defined by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \tag{7}$$

where $0 \leq \lambda < 1$ *and* $n \in N_0 = \{0, 1, 2, \cdots\}$.

Srivastava, Saigo and Owa defined the following fractional integral operator involving Gauss's hypergeometric function :

Definition 1.4 (Srivastava, Saigo and Owa[6]) *For real number* $\alpha > 0$, β *and* η, *the fractional integral operator* $I_{0,z}^{\alpha,\beta,\eta}$ *is defined by*

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \tag{8}$$

where f(z) is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), \quad \text{as } z \to 0,$$

where $\epsilon > \max\{0, \beta-\eta\} - 1$ *and the many-valuedness of* $(z-\xi)^{\alpha-1}$ *is removed by requiring* $\log(z-\xi)$ *to be real when* $(z-\xi) > 0$.

From Definition 1.1 and Definition 1.4, it is easy to see that

$$D_z^{-\alpha} f(z) = I_{0,z}^{\alpha,-\alpha,\eta} f(z).$$

Lemma 1.1 (Srivastava, Saigo and Owa[6]) *If* $\alpha > 0$ *and* $k > \beta - \eta - 1$, *then*

$$I_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}.$$

Using Definition 1.4 and Lemma 1.1 above, we showed the following result:

Theorem 1.1 ([5]) *Let* α, β *and* η *satisfy the inequalities,* $\alpha > 0$, $\beta < p+1$, $\alpha+\eta > -(p+1)$, $\beta - \eta < p+1$. *Choose a positive integer n such that*

$$n \geq \frac{\beta(\alpha+\eta)}{\alpha} - p - 1.$$

If $f(z) \in A(n,p,\{B_k\})$ *and* $B_k \geq B_{k+1}$, *then*

$$\left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq \max\left[0, \frac{\Gamma(p+1)\Gamma(p-\beta+\eta+1)}{\Gamma(p-\beta+1)\Gamma(p+\alpha+\eta+1)}|z|^{p-\beta} \times \left\{1-\frac{(p+1-\beta+\eta)_n(p+1)_n}{(p+1-\beta)_n(p+1+\alpha+\eta)_n B_{n+p}}|z|^n\right\}\right] \quad (9)$$

and

$$\left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq \frac{\Gamma(p+1)\Gamma(p-\beta+\eta+1)}{\Gamma(p-\beta+1)\Gamma(p+\alpha+\eta+1)}|z|^{p-\beta} \times \left\{1+\frac{(p+1-\beta+\eta)_n(p+1)_n}{(p+1-\beta)_n(p+1+\alpha+\eta)_n B_{n+p}}|z|^n\right\} \quad (10)$$

for $z \in U_0$, *where*

$$U_0 = \begin{cases} U, & \beta \leq p, \\ U-\{0\}, & \beta > p \end{cases}$$

and $(\lambda)_n$ *is the pochhammer symbol defied by* (4).

Equalities fold for the function defined by

$$f(z) = z^p - \frac{1}{B_{n+p}} z^{n+p}.$$

Recently, Choi, Kim, and Srivastava defined the following generalized operator of fractinal calculus:

Definition 1. 5 (Choi, Kim and Srivastava[1]) *Let* $\alpha, m \in R+$, *and* $\beta, \eta \in R$. *Then the fractional integral operator* $I_{0,z:m}^{\alpha,\beta,\eta}$ *is defined by*

$$I_{0,z:m}^{\alpha,\beta,\eta} f(z) = \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m - \xi^m)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\xi^m}{z^m}\right) f(\xi) d(\xi^m) \quad (11)$$

where the function ${}_2F_1$ *is Gauss's hypergeometric function defined by* (3) *with* $p-1=q=1$. *and f(z) is an analytic function in a simply-connected region of the z-plane containing the origin with the order*

$$f(z) = O(|z|^r), \quad z \to 0$$

where $r > max\{0, m(\beta-\eta)\} - m$, *and the multiplicity of* $(z^m - \zeta^m)^{\alpha-1}$ *is removed by requiring by* $\log(z^m - \zeta^m)$ *to be real when* $(z^m - \zeta^m) > 0$.

It is easy to observe that

$$I_{0,z;1}^{\alpha,-\alpha,\eta} f(z) = I_{0,z}^{\alpha,-\alpha,\eta} f(z).$$

We need the following Lemma:

Lemma 1. 2 *If* $\alpha > 0$ *and* $\dfrac{k}{m} > \beta - \eta - 1$, *then*

$$I_{0,z;m}^{\alpha,\beta,\eta} z^k = \frac{\Gamma\left(\dfrac{k}{m}+1\right)\Gamma\left(\dfrac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\dfrac{k}{m}+1-\beta\right)\Gamma\left(\dfrac{k}{m}+1+\alpha+\eta\right)} z^{k-m\beta}. \tag{12}$$

Proof. We shall prove Lemma 1.2 in a simillar fashion to the proof of Lemma 1.1 by Srivastava, Saigo and Owa[6].

$$\begin{aligned}
& I_{0,z;m}^{\alpha,\beta,\eta} z^k \\
&= \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m - \xi)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\xi^m}{z^m}\right) \xi^k d(\xi^m) \\
&= \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m - h)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{h}{z^m}\right) h^{\frac{k}{m}} dh \qquad (13) \\
&= \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\frac{k}{m}} {}_2F_1(\alpha+\beta, -\eta; \alpha; t)\, dt \\
&= \frac{z^{k-\beta}\Gamma\left(\dfrac{k}{m}+1\right)}{\Gamma\left(\dfrac{k}{m}+\alpha+1\right)} F\left(\alpha+\beta, -\eta; \frac{k}{m}\alpha+1; 1\right) \\
&= \frac{\Gamma\left(\dfrac{k}{m}+1\right)}{\Gamma\left(\dfrac{k}{m}+\alpha+1\right)} z^{k-m\beta} \frac{\Gamma\left(\dfrac{k}{m}+\alpha+1\right)\Gamma\left(\dfrac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\dfrac{k}{m}+1-\beta\right)\Gamma\left(\dfrac{k}{m}+1+\alpha+\eta\right)} \\
&= \frac{\Gamma\left(\dfrac{k}{m}+1\right)\Gamma\left(\dfrac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\dfrac{k}{m}+1-\beta\right)\Gamma\left(\dfrac{k}{m}+1+\alpha+eta\right)} z^{k-m\beta}. \qquad (14)
\end{aligned}$$

2 Distortion theorem

Theorem 2. 1 *Let $m \in N$ and α, β and η satisfy the inequalities $\alpha > 0$, $\beta < \frac{p}{m} + 1$, $\alpha + \eta > -\left(\frac{p}{m} + 1\right)$ and $\beta - \eta < \frac{p}{m} + 1$. Choose a positive integer n such that*

$$n \geq \frac{m\beta(\alpha + \eta)}{\alpha} - m - p.$$

If $f(z) \in A(n, p; \{B_k\})$ and $B_k \leq B_{k+1}$, then

$$\left|I_{0,z;m}^{\alpha,\beta,\eta} f(z)\right| \geq \max\left[0, \frac{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)}{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}|z|^{p-m\beta} \times \left\{1 - \frac{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)} \frac{\delta_m}{B_{n+p}}|z|^n\right\}\right] \tag{15}$$

and

$$\left|I_{0,z;m}^{\alpha,\beta,\eta} f(z)\right| \leq \frac{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)}{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}|z|^{p-m\beta} \times \left\{1 + \frac{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)} \frac{\delta_m}{B_{n+p}}|z|^n\right\} \tag{16}$$

for $z \in U_0$ where

$$U_0 = \begin{cases} U, & m\beta \leq p, \\ U - \{0\}, & m\beta > p \end{cases}$$

and δ_m is given by

$$\delta_m = \max_{n+p \leq k \leq n+p+m-1} \{\psi(k)\},$$

where

$$\psi(k) = \frac{\Gamma\left(\frac{k}{m}+1\right)\Gamma\left(\frac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\frac{k}{m}+1-\beta\right)\Gamma\left(\frac{k}{m}+1+\alpha+\eta\right)}.$$

Proof. Define a function $\Psi(z)$ by

$$\Psi(z)=\frac{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)}z^{m\beta}I_{0,z;m}^{\alpha,\beta,\eta}f(z). \tag{17}$$

Then, by virtue of Lemma 1.2, we have

$$\begin{aligned}\Psi(z)&\\ =&\ z^p-\sum_{k=n+p}^{\infty}\frac{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)}\\ &\times\frac{\Gamma\left(\frac{k}{m}+1\right)\Gamma\left(\frac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\frac{k}{m}+1-\beta\right)\Gamma\left(\frac{k}{m}+1+\alpha+\eta\right)}a_kz^k\\ =&\ z^p-\sum_{k=n+p}^{\infty}\Phi\psi(k)a_kz^k \end{aligned}\tag{18}$$

where

$$\Phi=\frac{\Gamma\left(\frac{p}{m}+1-\beta\right)\Gamma\left(\frac{p}{m}+1+\alpha+\eta\right)}{\Gamma\left(\frac{p}{m}+1\right)\Gamma\left(\frac{p}{m}+1-\beta+\eta\right)}$$

$$\psi(k)=\frac{\Gamma\left(\frac{k}{m}+1\right)\Gamma\left(\frac{k}{m}+1-\beta+\eta\right)}{\Gamma\left(\frac{k}{m}+1-\beta\right)\Gamma\left(\frac{k}{m}+1+\alpha+\eta\right)}.$$

For a positive integer n such that

$$n\geq\frac{m\beta(\alpha+\eta)}{\alpha}-m-p,$$

we have

$$0<\psi(k+m)\leq\psi(k)\quad(k\geq n+p).$$

Hence, defining δ_m by

$$\delta_m=\max\{\psi(n+p),\psi(n+p+1),\psi(n+p+2),\cdots,\psi(n+p+m-1)\},$$

we have

$$0 < \psi(k) \leq \delta_m \quad (k \geq n+p).$$

Therefore, we have that

$$\begin{aligned} |\Phi(z)| &\leq |z|^p + \sum_{k=n+p}^{\infty} \Phi\Psi(k) a_k |z|^k \\ &\leq |z|^p + \Phi\delta_m |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ &\leq |z|^p + \Phi\delta_m |z|^{n+p} \frac{1}{B_{n+p}}. \end{aligned} \tag{19}$$

We have the last inequality above, because

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{1}{B_{n+p}}$$

by the assumption of the theorem such that $f(z) \in A(n, p; \{B_k\})$ and $B_k \leq B_{k+1}$.

In the same manner, we have

$$|\Phi(z)| \geq \max \left\{ 0, |z|^p - \Phi\delta_m |z|^{n+p} \frac{1}{B_{n+p}} \right\}. \tag{20}$$

By virtue of (17), these estimates (19) and (20) lead to (15) and (16), respectively.

Remark 2. 1 *If* $m = 1$ *in Theorem 2.1, then we have Theorem 1.1*([5]).

References

[1] J. H. Choi, Y . C. Kim and H. M. Srivastava, Starlike and convexity of fractional calculus operators, in preparation.

[2] S. Owa, On the distortion theorems, Kyungpook Math. J. 18(1978), 53-59.

[3] S. Owa and M. Obradovic, New classification of analytic functions with negative coefficients, Intern. J. Math. and Math. Sci. (1988), 55-70.

[4] T. Sekine, On new generalized classes of analytic functions with negative coefficients, Report Res. Inst. Sci. Tec. Nihon Univ. 35(1987),1-26

[5] T. Sekine, Distortion theorems for fractional calculus on a generalized subclasses of p-valent functions, in Proceedings of the First Korea-Japanese Colloquium on Finite or Infinite Dimentional Complex Analysis(J.Kajiwara, H.Kazama and K.H.Shon ed.), Pusan, 1993, 141-149.

[6] H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Aappl. 131(1988), 421-440.