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An opinion on non-Imaginary part of Gamma-Starlike Functions

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1 Introduction.

Let A denote the class of functions $f(z)$ analytic in $E = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

A function $f(z) \in A$ is called starlike with respect to the origin if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \qquad in \quad E,$$

and a function $f(z) \in A$ is said to be convex if and only if

$$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} > 0. \qquad in \quad E.$$

In [1], Lewandowski,Miller and Zlotkiewicz defined Gamma-Starlike Function as the following.

Definition. Let $f(z) \in A$ and suppose that $f(z) \neq 0$, $f'(z) \neq 0$, and $1 + \frac{zf''(z)}{f'(z)} \neq 0$ in $0 < |z| < 1$.

Suppose γ is a real number and

(1)
$$\mathrm{Re}(\frac{zf'(z)}{f(z)})^{1-\gamma}(1 + \frac{zf''(z)}{f'(z)})^{\gamma} > 0$$

for $z \in E$, where the power appearing in (1) are meant as principal values.

If $f(z) \in A$ satisfies the condition (1), then we say that $f(z)$ is a gamma-starlike function and we denote the class of such functions by L_γ.

Remarks. (i) Condition (1) is equivalent to the following condition

$$|(1-\gamma)arg\frac{zf'(z)}{f(z)} + \gamma arg(1 + \frac{zf''(z)}{f'(z)})| < \frac{\pi}{2}.$$

(ii) If $\gamma = 0$, $L_0 \equiv S^*$, the class of starlike functions, while if $\gamma = 1$,$L_1 \equiv C$, the class of convex functions.

In [1], they obtained the following result.

Theorem A. $L(\gamma) \subset S^*$, for all real γ .

Let N be the class of functions $p(z)$ analytic in E and $p(0) = 1$. We call $p(z) \in N$ a Carathéodory function, if it satisfies the condition $\mathrm{Re}p(z) > 0$ in E.

2 Preliminary.

In this paper, we need the following lemma.

Lemma [2]. Let $p(z) \in N$ and suppose that there exists a point $z_0 \in E$ such that $\mathrm{Re}p(z) > 0$ for $|z| < |z_0|$, and $\mathrm{Re}p(z_0) = 0$ $(p(z_0) \neq 0)$.
Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is a real number and

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \geq 1 \quad when \quad p(z_0) = ia,\ a > 0,$$

and

$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \leq -1 \quad when \quad p(z_0) = ia,\ a < 0.$$

3 Main result.

Now,we prove the following Theorem.

Theorem. Let $f(z) \in A$ and let $f(z) \neq 0$, $f'(z) \neq 0$ and $1 + \frac{zf''(z)}{f'(z)} \neq 0$ in $0 < |z| < 1$. Suppose that $(\frac{zf'(z)}{f(z)})^{1-\gamma}(1 + \frac{zf''(z)}{f'(z)})^{\gamma} \neq il$ in E, where $\gamma \geq \frac{1}{2}$, l is a real number and

$$|l| > \begin{cases} \sqrt{(\frac{2\gamma-1}{3})}(\frac{3\gamma}{2\gamma-1})^{\gamma}, & \gamma > \frac{1}{2}, \\ \frac{\sqrt{2}}{2}, & \gamma = \frac{1}{2}, \end{cases}$$

and for the case $\gamma < \frac{1}{2}$, suppose that

(2) $$\mathrm{Re}(\frac{zf'(z)}{f(z)})^{1-\gamma}(1 + \frac{zf''(z)}{f'(z)})^{\gamma} > 0 \qquad in \quad E.$$

Then $f(z)$ is starlike in E.

Proof. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}.$$

If there exists a point $z_0 \in E$ such that $\mathrm{Re}p(z)>0$ for $|z|<|z_0|$, and $\mathrm{Re}p(z_0)=0$ ($p(z_0)\neq 0$), then from Lemma, we have

$$\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)^{\gamma}=(p(z_0))^{1-\gamma}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)^{\gamma}.$$

Therefore we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)^{\gamma} &= \mathrm{Re}\ (ia)^{1-\gamma}(ia+ik)^{\gamma}\\ &= \mathrm{Re}\ ia\left(1+\frac{k}{a}\right)^{\gamma}\\ &= 0\end{aligned}$$

where $p(z_0)=ia$ (a is a real number) and from Lemma, a and k are the same sign.

For the case $a>0$, let us put

$$g(a)=a\left(1+\frac{k}{a}\right)^{\gamma}.$$

From Lemma, we have

$$g(a)\geq a\left(\frac{3}{2}+\frac{1}{2a^2}\right)^{\gamma},\qquad (r\geq 0). \tag{3}$$

Putting $q(a)$ the last term of (3), let us get the mininum value m of $q(a)$ for $a>0$.

Differentiation $q(a)$, we have

$$q'(a)=\left(\frac{3}{2}+\frac{1}{2a^2}\right)^{\gamma-1}\left(\frac{3}{2}+\frac{1}{2a^2}-\frac{\gamma}{a^2}\right). \tag{4}$$

Since we have

$$\left(\frac{3}{2}+\frac{1}{2a^2}\right)^{\gamma-1}>0,$$

and so $q'(a)$ become 0 only at $a=\sqrt{\frac{2\gamma-1}{3}}$.

Therefore, for the case $\gamma>\frac{1}{2}$, $q(a)$ takes its mininum value m at $a=\sqrt{\frac{2\gamma-1}{3}}$,and

$$m=q\left(\sqrt{\frac{2\gamma-1}{3}}\right)=\sqrt{\frac{2\gamma-1}{3}}\left(\frac{3\gamma}{2\gamma-1}\right)^{\gamma},$$

and for the case $\gamma=\frac{1}{2}$, $q(a)$ takes its mininum value m at $a=0$,and

$$m=\lim_{a\to+0}q(a)=\lim_{a\to+0}a\left(\frac{3}{2}+\frac{1}{2a^2}\right)^{\frac{1}{2}}=\frac{\sqrt{2}}{2}.$$

These contradict (2).

On the other hand, if there exists a point $z_0\in E$ such that $\mathrm{Re}p(z)>0$ for $|z|<|z_0|$, $\mathrm{Re}p(z_0)=0$ ($p(z_0)\neq 0$) and $p(z_0)=ia$, $a<0$.

Applying the same method as the above,we have

$$q(a) \le \begin{cases} -\sqrt{(\frac{2\gamma-1}{3})}(\frac{3\gamma}{2\gamma-1})^{\gamma}, & \gamma > \frac{1}{2}, \\ -\frac{\sqrt{2}}{2}, & \gamma = \frac{1}{2}, \end{cases}$$

These also contradict (2).

For the case $\gamma < \frac{1}{2}$, if these exists a point $z_0 \in E$ such that $\mathrm{Re}p(z) > 0$ for $|z| < |z_0|$ and $\mathrm{Re}p(z_0) = 0$ $(p(z_0) \neq 0)$.

Then from Lemma, we have

$$\begin{aligned} \mathrm{Re}(\frac{z_0 f'(z_0)}{f(z_0)})^{1-\gamma}(1+\frac{z_0 f''(z_0)}{f'(z_0)})^{\gamma} &= \mathrm{Re}(p(z_0))^{1-\gamma}(p(z_0)+\frac{z_0 p'(z_0)}{p(z_0)})^{\gamma} \\ &= 0. \end{aligned}$$

This contradicts (2).

Therefore we have $\mathrm{Re}p(z) > 0$ in E, or $f(z)$ is starlike in E.

From Main theorem we easily have Theorem A, and so this theorem completely improved Theorem A [1].

Further, letting $\gamma = 1$ in Main theorem, we easily have

Corollary [3]. If $f(z) \in A(1)$ and

$$|\mathrm{Im}\frac{zf''(z)}{f'(z)}| < \sqrt{3} \qquad in \quad E,$$

then f(z) is univalently starlike in E.

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