数理解析研究所講究録
963 巻 1996 年 55-62

INCLUSION PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

LI JIAN LIN（西北工業大学）
SHIGEYOSHI OWA（近畿大学・理工）

ABSTRACT. The object of the present paper is to give sharp forms of inclusion properties of the class $P(p,\alpha,\beta)$ under operators $J_{p,c}$, F_m and $J^{\lambda}_{p,1}$.

I. INTRODUCTION

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be a member of the class $P(p,\alpha)$ if it satisfies the inequality $\mathrm{Re}\{f'(z)/z^{p-1}\} > \alpha$ $(z \in U)$ for some α $(0 \leqq \alpha < p)$. The classes $P(1,0)$ and $P(p,0)$ were investigated by MacGregor [5] and Umezawa [6], respectively.

For a function $f(z)$ belonging to $A(p)$, we define the generalized Bernardi integral operator $J_{p,c}$ by

$$J_{p,c}(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt$$

$$= z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} a_{p+n} z^{p+n} \qquad (p \in N;\ c > -p). \tag{1.2}$$

The operator $J_{1,c}$ for $c \in N$ was introduced by Bernardi [1]. Clearly, from (1.2) we see that

$$f(z) \in A(p) \Longrightarrow J_{p,c}(f) \in A(p) \qquad (c > -p). \tag{1.3}$$

1991 Mathematics Subject Classification. Primary 30C45.

Thus, by applying the operator $J_{p,c}$ successively, we can obtain

$$J^n_{p,c}(f) = \begin{cases} J_{p,c}(J^{n-1}_{p,c}(f)) & (n \in \mathbb{N}) \\ f(z) & (n = 0). \end{cases} \tag{1.4}$$

Suppose also that

$$F_m(f) = J_{p,c_m}(J_{p,c_{m-1}} \cdots (J_{p,c_1}(f)))$$

$$= z^p + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{m} \frac{p+c_j}{p+n+c_j} \right) a_{p+n} z^{p+n} \qquad (c_j > -p;\ m \in \mathbb{N}). \tag{1.5}$$

For an analytic function $g(z)$ given by $g(z) = \sum_{n=0}^{\infty} b_{p+n} z^{p+n}$ in $\mathbb{U}$, and for a real number λ, Flett [3] define the multiplier transformation $I^\lambda g(z)$ by

$$I^\lambda g(z) = \sum_{n=0}^{\infty} (p+n+1)^{-\lambda} b_{p+n} z^{p+n} \qquad (z \in \mathbb{U}). \tag{1.6}$$

The function $I^\lambda g(z)$ is clearly analytic in $\mathbb{U}$. It may be regarded as a fractional integral (for $\lambda > 0$) or a fractional derivative (for $\lambda < 0$) of $g(z)$. Furthermore, in terms of the Gamma function, we have

$$I^\lambda g(z) = \frac{1}{\Gamma(\lambda)} \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} g(tz)\,dt \qquad (\lambda > 0). \tag{1.7}$$

Denote by $D^\lambda g(z)$ the multiplier transfomation $I^{-\lambda} g(z)$ for $\lambda \geqq 0$, i.e.,

$$D^\lambda g(z) = I^{-\lambda} g(z) = \sum_{n=0}^{\infty} (p+n+1)^{\lambda} b_{p+n} z^{p+n} \qquad (\lambda \geqq 0;\ z \in \mathbb{U}). \tag{1.8}$$

From (1.4) and (1.6),

$$J^m_{p,1}(f) = (p+1)^m I^m(f) \qquad (m \in \mathbb{N};\ f \in \mathcal{A}(p)). \tag{1.9}$$

Thus, one can define the operator $J^\lambda_{p,1}$ (depending on a continuous parameter $\lambda > 0$) by

$$J^\lambda_{p,1}(f) = (p+1)^\lambda I^\lambda(f) \qquad (\lambda > 0;\ f \in \mathcal{A}(p)). \tag{1.10}$$

Making use of the fractional derivative operator and operators $J_{p,c}$, F_m and $J^{\lambda}_{p,1}$ as mentioned above, Cho [2] introduced and studied the class $\mathcal{P}(p,\alpha,\beta)$ defined by

$$\mathcal{P}(p,\alpha,\beta) = \{f \in \mathcal{A}(p): (p+1)^{-\beta}D^{\beta}f \in \mathcal{P}(p,\alpha)\},$$

where $0 \leq \alpha < p$ and $\beta \geq 0$. Observe that $\mathcal{P}(p,\alpha,0) = \mathcal{P}(p,\alpha)$. If $\beta \geq 0$ and $0 \leq \alpha_1 \leq \alpha_2 < p$, then $\mathcal{P}(p,\alpha_2,\beta) \subset \mathcal{P}(p,\alpha_1,\beta)$. The class $\mathcal{P}(1,\alpha,\beta)$ was introduced and studied by Kim, Lee and Srivastava [4]. In [2], Cho showed that

(i) if $f(z) \in \mathcal{P}(p,\alpha,\beta)$, then $J_{p,c}(f)$, $F_m(f)$ and $J^{\lambda}_{p,1}(f)$ are slao in the class $\mathcal{P}(p,\alpha,\beta)$, where $c \in \mathbb{N}$ and $c_j \in \mathbb{N}$,

(ii) if $0 \leq \alpha < p$ and $\beta \geq 0$, then $\mathcal{P}(p,\alpha,\beta+1) \subset \mathcal{P}(p,\mu,\beta)$, where $\mu = (2\alpha(p+1)+p)/(2(p+1)+1)$.

In the case of $p = 1$, these results correspond to the results by Kim, Lee and Srivastava [4]. In the present paper, we give the sharp forms of these results simply.

2. Inclusion Properties

Our first result for the class $\mathcal{P}(p,\alpha,\beta)$ is contained in

THEOREM I. If $f(z)$ is in the class $\mathcal{P}(p,\alpha,\beta)$, then $J_{p,c}(f)$ belongs to the class $\mathcal{P}(p,\mu,\beta)$, where

$$\mu = p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c} .$$

The result is sharp.

PROOF. It follows from the definitions (1.2) and (1.8) that

$$(p+1)^{-\beta}D^{\beta}(J_{p,c}(f)) = J_{p,c}((p+1)^{-\beta}D^{\beta}(f))$$

$$= (p+c) \int_0^1 t^{c-1}\{(p+1)^{-\beta}D^{\beta}f(tz)\}dt. \tag{2.1}$$

Therefore, setting

$$H(z) = (p+1)^{-\beta} D^{\beta}(J_{p,c}(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta} D^{\beta} f(z), \tag{2.2}$$

we must show that

$$\mathrm{Re}\left(\frac{H'(z)}{z^{p-1}}\right) > \mu \qquad (0 \leqq \alpha < p;\ c > -p;\ z \in \mathcal{U}) \tag{2.3}$$

whenever $h(z) \in \mathcal{P}(p,\alpha)$. Note that (2.1) gives

$$\mathrm{Re}\left(\frac{H'(z)}{z^{p-1}}\right) = (p+c) \int_0^1 t^{p+c-1}\, \mathrm{Re}\left(\frac{h'(tz)}{(tz)^{p-1}}\right) dt. \tag{2.4}$$

Since $h(z) \in \mathcal{P}(p,\alpha)$, we have

$$\mathrm{Re}\left(\frac{h'(tz)}{(zt)^{p-1}}\right) > \frac{p-(p-2\alpha)t}{1+t} \qquad (0 < t \leqq 1;\ z \in \mathcal{U}) \tag{2.5}$$

and hence (2.4) yields

$$\mathrm{Re}\left(\frac{H'(z)}{z^{p-1}}\right) > (p+c) \int_0^1 t^{p+c-1} \frac{p-(p-2\alpha)t}{1+t}\, dt$$

$$= p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c}\ . \tag{2.6}$$

Further, to show that the result is sharp, we consider the function

$$f_0(z) = z^p + \sum_{n=1}^{\infty} \frac{2(p-\alpha)(p+1)^{\beta}}{(p+n)(p+n+1)^{\beta}} (-1)^n z^{p+n}\ , \tag{2.7}$$

which belongs to the class $\mathcal{P}(p,\alpha,\beta)$. Since

$$H_0(z) = (p+1)^{\beta} D^{\beta}(J_{p,c}(f_0(z)))$$

$$= z^p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{(p+n)(p+n+c)} z^{p+n} \qquad (c > -p),$$

one can easily show that $J_{p,c}(f_0(z)) \in \mathcal{P}(p,\mu,\beta)$, but $J_{p,c}(f_0(z)) \notin \mathcal{P}(p,\mu',\beta)$ if $\mu' > \mu$. This completes the proof of Theorem 1.

REMARK 1. For $0 \leqq \alpha < p$ and $c > -p$, we have from the right-hand side of (2.6) that $\alpha \leqq \mu < p$, and hence $\mathcal{P}(p,\mu,\beta) \subset \mathcal{P}(p,\alpha,\beta)$ $(\beta \geqq 0)$.

COROLLARY 1. If $f(z) \in P(p,\alpha,\beta)$, then $F_m(f(z))$ defined by (1.5) belongs to the class $P(p,\mu_m,\beta)$, where

$$\mu_j = p + 2(p-\mu_{j-1})(p+c_j) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c_j} \qquad (j = 1,2,3,\ldots,m)$$

and $\mu_0 = \alpha$. The result is sharp.

Next, we derive

THEOREM 2. If $f(z)$ is in the class $P(p,\alpha,\beta)$, then $J^{\lambda}_{p,1}(f(z))$ defined by (1.10) belongs to the class $P(p,\gamma,\beta)$, where

$$\gamma = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+1}{p+n+1}\right)^{\lambda} .$$

The result is sharp.

PROOF. Making use of (1.7) and (1.8), the definition (1.10) yields

$$(p+1)^{-\beta}D^{\beta}(J^{\lambda}_{p,1}(f(z))) = J^{\lambda}_{p,1}((p+1)^{-\beta}D^{\beta}(f(z)))$$

$$= \frac{(p+1)^{\lambda}}{\Gamma(\lambda)} \int_0^1 \left(\log\frac{1}{t}\right)^{\lambda-1} (p+1)^{-\beta}D^{\beta}(f(tz))dt \qquad (\lambda > 0;\ \beta \geqq 0) \tag{2.8}$$

Therefore, setting

$$G(z) = (p+1)^{-\beta}D^{\beta}(J^{\lambda}_{p,1}(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta}D^{\beta}(f(z)), \tag{2.9}$$

we have to show that

$$\mathrm{Re}\left(\frac{G'(z)}{z^{p-1}}\right) > \gamma \qquad (z \in U) \tag{2.10}$$

whenever $h(z) \in P(p,\alpha)$. Applying (2.5), we obtain

$$\mathrm{Re}\left(\frac{G'(z)}{z^{p-1}}\right) = \frac{(p+1)^{\lambda}}{\Gamma(\lambda)} \int_0^1 \left(\log\frac{1}{t}\right)^{\lambda-1} t^p \mathrm{Re}\left(\frac{h'(tz)}{(tz)^{p-1}}\right) dt$$

$$> \frac{(p+1)^{\lambda}}{\Gamma(\lambda)} \int_0^1 \left(\log\frac{1}{t}\right)^{\lambda-1} t^p \frac{p - (p-2\alpha)t}{1 + t} dt$$

$$= p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+1}{p+n+1}\right)^{\lambda} . \tag{2.11}$$

To show that the result is sharp, we take the function $f_0(z)$ given by (2.7). Since

$$G_0(z) = (p+1)^{-\beta} D^{\beta} (J_{p,1}^{\lambda}(f_0(z)))$$

$$= z^p + 2(p-\alpha) \sum_{n=1}^{\infty} \left(\frac{p+1}{p+n+1}\right)^{\lambda} \frac{(-1)^n}{p+n} z^{p+n} \tag{2.12}$$

with $0 \leqq \alpha < p$, $\lambda > 0$ and $\beta \geqq 0$, we see that $J_{p,1}^{\lambda}(f_0(z)) \in P(p,\gamma,\beta)$, but $J_{p,1}^{\lambda}(f_0(z)) \notin P(p,\gamma',\beta)$ if $\gamma' > \gamma$. Thus we complete the proof of the theorem.

REMARK 2. For $0 \leqq \alpha < p$ and $\lambda > 0$, we have from the right-hand side of (2.11) that $\alpha \leqq \gamma < p$, and hence $P(p,\gamma,\beta) \subset P(p,\alpha,\beta)$ $(\beta \geqq 0)$.

COROLLARY 2. If $0 \leqq \alpha < p$ and $0 \leqq \beta < \rho$, then $P(p,\alpha,\rho) \subset P(p,\gamma_0,\beta)$, where

$$\gamma_0 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+n}{p+n+1}\right)^{\rho-\beta} .$$

The result is sharp.

PROOF. Setting $\lambda = \rho-\beta > 0$ in Theorem 2, we observe that

$$\begin{aligned} f(z) \in P(p,\alpha,\rho) &\Longrightarrow J_{p,1}^{\rho-\beta}(f(z)) \in P(p,\gamma_0,\rho) \\ &\Longleftrightarrow (p+1)^{-\rho} D^{\rho} (J_{p,1}^{\rho-\beta}(f(z))) \in P(p,\gamma_0) \\ &\Longleftrightarrow (p+1)^{-\beta} D^{\beta}(f(z)) \in P(p,\gamma_0) \\ &\Longleftrightarrow f(z) \in P(p,\gamma_0,\beta). \end{aligned} \tag{2.13}$$

COROLLARY 3. If $0 \leqq \alpha < p$ and $\beta \geqq 0$, then $P(p,\alpha,\beta+1) \subset P(p,\gamma_1,\beta)$, where

$$\gamma_1 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \frac{p+1}{p+n+1} .$$

The result is sharp.

PROOF. Putting $\lambda = 1$ in Theorem 2, we have

$$f(z) \in P(p,\alpha,\beta+1) \Longrightarrow J^1_{p,1}(f(z)) \in P(p,\gamma_1,\beta+1)$$

$$\Longleftrightarrow (p+1)^{-\beta-1}D^{\beta+1}(J^1_{p,1}(f(z))) \in P(p,\gamma_1)$$

$$\Longleftrightarrow (p+1)^{-\beta}D^{\beta}(f(z)) \in P(p,\gamma_1)$$

$$\Longleftrightarrow f(z) \in P(p,\gamma_1,\beta).$$

REMARK 3. We note that the several cases of Theorem 1 and Theorem 2, for the special values of p, α, β, c and λ, will improve some known results.

ACKNOWLEDGMENTS

The present investigation of the second author was supported. in part, by the Japanese Ministry of Education, Science and Culture under a grant-in-aid for general scientific research.

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Department of Applied Mathematics
Northwestern Polytechnical University
Xi An, Shaan Xi 710072
People's Republic of China

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan