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ON CERTAIN MEROMORPHIC P-VALENT FUNCTIONS

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ABSTRACT. A certain differential operator D^n is introduced for functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}a_kz^k$$

which are analytic in $E^*=\{z:0<|z|<1\}$. The object of the present paper is to give an application of the above operator D^n to the differential inequalities.

Keywords. Analytic, p-valent, meromorphic.

1. INTRODUCTION

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}a_kz^k \quad (p\in\mathbb{N}=\{1,2,\cdots\})$$

which are analytic in $E^*=\{z:0<|z|<1\}$. Define

$$D^0f(z)=f(z)$$
$$D^1f(z)=\frac{1}{z^p}+(p+1)a_0+(p+2)a_1z+(p+3)a_2z^2+\cdots$$
$$D^2f(z)=D(D^1f(z))$$

and for $n=1,2,\cdots$

$$D^nf(z)=D(D^{n-1}f(z))=\frac{1}{z^p}+\sum_{m=1}^{\infty}(p+m)^na_{m-1}z^{m-1}.$$

Recently Uralegaddi and Somanatha [1] and Aouf and Hossen [2] have studied certain class of meromorphic multivalent functions defined by the operator $D^nf(z)$. The object of the present paper is to investigate some new properties of meromorphic p-valent functions defined by the above operator.

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Definition. Let H be the set of complex valued functions $h(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$ ($\mathbb{C}$ is the complex plane) such that

$$h(r,s,t) \text{ is continuous in a domain } D \subset \mathbb{C}^3; \tag{1.1}$$

$$(1,1,1) \in D \text{ and } |h(1,1,1)| < 1; \tag{1.2}$$

$$\left|h(e^{i\theta}, m+e^{i\theta}, \frac{m+L+3me^{i\theta}+e^{2i\theta}}{m+e^{i\theta}})\right| \geqslant 1, \tag{1.3}$$

whenever

$$(e^{i\theta}, m+e^{i\theta}, \frac{m+L+3me^{i\theta}+e^{2i\theta}}{m+e^{i\theta}}) \in D$$

with $\mathrm{Re}L \geqslant m(m-1)$ for real θ and for real $m \geqslant 1$.

2. Main Result

In proving our main result, we shall need the following lemma due to Miller and Mocanu [3].

Lemma. Let $w(z) = a + w_k z^k + \cdots$ be analytic in $E = \{z : |z| < 1\}$ with $w(z) \not\equiv a$ and $k \geqslant 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leqslant r_0} |w(z)|$, then

$$z_0 w'(z_0) = m w(z_0) \quad \text{and} \tag{2.1}$$

$$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geqslant m, \tag{2.2}$$

where m is real and

$$m \geqslant k\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2} \geqslant k\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

Theorem. *Let $h(r,s,t) \in H$ and let $f(z)$ belonging to $\Sigma(p)$ satisfy*

$$(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)}) \in D \subset \mathbb{C}^3 \quad \textit{and} \tag{2.3}$$

$$\left|h(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)})\right| < 1 \tag{2.4}$$

for all $z \in E$. Then we have

$$\left|\frac{D^n f(z)}{D^{n-1} f(z)}\right| < 1 \quad (z \in E).$$

Proof. Let

$$\frac{D^n f(z)}{D^{n-1} f(z)} = w(z),$$

then it follows that $w(z)$ is either analytic or meromorphic in E, $w(0) = 1$ and $w(z) \not\equiv 1$. With the aid of the identity (easy to verify)

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z),$$

we obtain

$$\frac{D^{n+1} f(z)}{D^n f(z)} = w(z) + \frac{zw'(z)}{w(z)}$$

$$\frac{D^{n+2} f(z)}{D^{n+1} f(z)} = w(z) + \frac{zw'(z)}{w(z)} + \frac{zw'(z) + \frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - (\frac{zw'(z)}{w(z)})^2}{w(z) + \frac{zw'(z)}{w(z)}}$$

we claim that $|w(z)| < 1$ for $z \in E$. Otherwise there exists a point $z_0 \in E$ such that $\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using lemma with $a = 1$ and $k = 1$, we see that

$$\frac{D^n f(z_0)}{D^{n-1} f(z_0)} = e^{i\theta},$$

$$\frac{D^{n+1} f(z_0)}{D^n f(z_0)} = m + e^{i\theta},$$

$$\frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} = \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}},$$

where $L = \dfrac{z_0^2 w''(z_0)}{w(z_0)}$ and $m \geqslant 1$.

Further, an application of (2.2) in lemma gives

$$\text{Re} L \geqslant m(m-1).$$

Since $h(r, s, t) \in H$, we have

$$\left| h(\frac{D^n f(z_0)}{D^{n-1} f(z_0)}, \frac{D^{n+1} f(z_0)}{D^n f(z_0)}, \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)}) \right|$$
$$= \left| h(e^{i\theta}, m + e^{i\theta}, \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}) \right|$$
$$\geqslant 1.$$

which contradicts the condition (2.4) of the theorem. Therefore, we conclude that

$$\left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1 \quad (z \in E).$$

This completes the proof of Theorem.

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