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A characterization of rational singularities in terms of injectivity of Frobenius maps

(Frobenius 写像の単射性による有理特異点の特徴付け)

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Introduction

In [HH1], Hochster and Huneke introduced the notion of the tight closure of an ideal in a ring of characteristic $p>0$. Tight closure enables us to define classes of rings of characteristic p such as F-rational rings [FW] and F-regular rings [HH1], and it turns out that they are closely related with some classes of singularities in characteristic 0 defined via resolution of singularity.

It was shown by Smith [S] that a ring in characteristic 0 has a rational singularity if its modulo p reduction is F-rational for infinitely many p, and Watanabe [W] obtained an analoguous result for log terminal singularity and F-regularity. The essential parts of these results hold true in *arbitrary* positive characteristic. But if we consider the converse implication, we soon confront some difficulty arising from pathological phenomena in small characteristic $p>0$. For example, a two-dimensional log terminal singularity is always F-regular for $p>5$, but is not F-regular in general if $p=2$, 3 or 5 [Ha]. To avoid such difficulty we will look at generic behavior of modulo p reduction for sufficiently large p. In this context, Fedder gave affirmative answer in some special cases [F1,2]: If a graded ring R over a field of characteristic 0 has a rational singularity and if R is a complete intersection or $\dim R=2$, then modulo p reduction of R is F-rational for $p\gg 0$.

Unfortunately, except for the above special cases, the implication "rational singularity $\Rightarrow$ F-rational" remained to be open, and is considered to be one of the fundamental problems in the tight closure theory. We aim to give an affirmative answer to this question in a fairly general situation.

A d-dimensional Cohen-Macaulay local ring (R,m) of characteristic $p>0$ is F-rational if and only if the tight closure $(0)^*$ of (0) in $H_m^d(R)$ coincides (0) itself. Let us assume that R has an isolated singularity and that there is a "good" resolution of singularity $f:X\to Y=\mathrm{Spec}R$, that is, a resolution with simple normal crossing exceptional divisor E (in characteristic 0, such a resolution exists [Hi]). If D is an f-ample fractional divisor such that $-D$ has no integral part, then we observe that R is F-rational if it has at most "rational" singularity (i.e., $H^i(X,\mathcal{O}_X)=0$ for $i>0$) and if the iterated Frobenius map

$$F^e: H_E^d(X,\mathcal{O}_X)\to H_E^d(X,\mathcal{O}_X(-qD))$$

is injective for all powers $q=p^e$. This is a generalization of Fedder and Watanabe's result for graded rings [FW].

To analize the above Frobenius maps we will use log de Rham complex $\Omega_X^\bullet(\log E)$ and the Cartier operator [C], [Ka]. We see that an obstruction for the map to be injective lies in non-vanishing of certain cohomology groups (3.2). However, a slight generalization of Deligne and Illusie's proof of the Akizuki-Kodaira-Nakano vanishing theorem for characteristic $p > 0$ [DI] and the Serre vanishing theorem imply that these cohomology groups vanish for $p \gg 0$ if we reduce X and D from characteristic 0 to characteristic $p > 0$.

Consequently, our argument establishes the correspondence of rational singularity with F-rationality (3.1), and also that of log terminal singularity with F-regularity (3.5).

1 Preliminaries

Let R denote a Noetherian ring. We will often assume that R has prime characteristic $p > 0$. In this case we always use the letter q for a power p^e of p. Also, R^0 will denote the set of elements of R which is not in any minimal prime ideal.

Definition (1.1) [HH1]. Let R be a Noetherian ring of characteristic $p > 0$, and $I \subset R$ be an ideal. The *tight closure* I^* of I in R is the ideal defined by $x \in I^*$ iff there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for $q = p^e \gg 0$, where $I^{[q]}$ is the ideal generated by the q-th powers of the elements of I. We say that I is *tightly closed* if $I^* = I$.

Definition (1.2). Let R denote a Noetherian ring of characteristic $p > 0$.

(i) [FW] A local ring (R, m) is said to be *F-rational* if some (or, equivalently, every) ideal generated by system of parameters of R is tightly closed. When R is not local, we say that R is F-rational if every localization is F-rational.

(ii) [HH1] R is said to be *F-regular* if every ideal of R is tightly closed.

Remark (1.2.1). In characteristic $p > 0$, the following implications are known [HH1]:

$$\text{regular} \Rightarrow F\text{-regular} \Rightarrow F\text{-rational} \Rightarrow \text{normal}.$$

Also, an F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay, and a Gorenstein F-rational ring is F-regular.

Definition (1.3). Let R be a Noetherian ring of characteristic $p > 0$. An element $c \in R^0$ is said to be a *test element* if for all ideals $I \subset R$ and $x \in R$, one has

$$x \in I^* \iff cx^q \in I^{[q]} \text{ for all } q = p^e \ (e \geq 0).$$

Proposition (1.4) [HH2]. If R is a reduced excellent local ring of characteristic $p > 0$, and $c \in R^0$ is an element such that R_c is regular, then some power of c is a test element for R.

Given a property P defined for rings of characteristic $p > 0$ such as "F-rational" or "F-regular", we will extend the concept to characteristic 0 using the technique of reduction modulo p.

Definition (1.5) (cf. [HR]). Let R be a finitely generated algebra over a field k of characteristic 0. We say that R is of *P type* if there exist a finitely generated $\mathbf{Z}$-subalgebra A of k and a finitely generated A-algebra R_A satisfying the following conditions:

(i) R_A is flat over A and $R_A \otimes_A k \cong R$.

(ii) $R_\kappa = R_A \otimes_A \kappa(s)$ has property P for every closed point s in a dense open subset of $S = \mathrm{Spec}A$, where $\kappa = \kappa(s)$ is the residue field of $s \in S$.

Remark (1.5.1). In condition (ii), the fiber ring $R_\kappa = R_A \otimes_A \kappa$ always has positive characteristic since A is finitely generated over $\mathbf{Z}$. We sometimes abbreviate the statement in condition (ii) as "R_κ has property P for general closed points $s \in S$ with residue field $\kappa = \kappa(s)$". However, if R is of P type, we can replace $S = \mathrm{Spec}A$ by a suitable open subset so that condition (ii) holds for *every* closed point $s \in S$.

A normal ring R in characteristic 0 is said to have rational singularity if for a resolution of singularity $f : X \to \mathrm{Spec}R$, one has $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. The aim of the present paper is to show the converse of the following result due to Smith [S].

Theorem (1.6) [S]. *Let R be a finitely generated algebra over a field of characteristic zero. If R is of F-rational type, then it has at most rational singularity.*

2 Log de Rham complex and the Cartier operator

We will review some fundamental facts about log de Rham complex and the Cartier operator in characteristic $p > 0$. Concerning these subjects the reader may consult [C] and [Ka] (see also [EV]).

Assumption (2.1). Throughout this section X will denote a d-dimensional *smooth* variety of finite type over a perfect field k of characteristic $p > 0$, and $E = \sum_{j=1}^m E_j$ a reduced simple normal crossing divisor on X, that is, a divisor with smooth irreducible components E_j intersecting transversally.

Let us choose local parameters $t_1, \ldots, t_d$ of X so that E is locally defined by $t_1 \cdots t_s = 0$. Then we can consider the locally free $\mathcal{O}_X$-module $\Omega_X^1(\log E)$ with local basis

$$\frac{\mathrm{d}t_1}{t_1}, \ldots, \frac{\mathrm{d}t_s}{t_s}, \mathrm{d}t_{s+1}, \ldots, \mathrm{d}t_d.$$

We define $\Omega_X^i(\log E) = \bigwedge^i \Omega_X^1(\log E)$ for $i \geq 0$. These sheaves, together with the differential maps d, give rise to a complex $\Omega_X^\bullet(\log E)$ called a log de Rham complex.

(2.2) *The Cartier operator* [C], [Ka]. Let $F : X \to X$ be the absolute Frobenius morphism of X. The direct image $F_*\Omega_X^\bullet(\log E)$ of the de Rham complex can be viewed as a complex of $\mathcal{O}_X$-modules via $F^* : \mathcal{O}_X \to F_*\mathcal{O}_X$. We denote the i-th cohomology sheaf of this complex by $\mathcal{H}^i(F_*\Omega_X^\bullet(\log E))$. Then, there is an isomorphism of $\mathcal{O}_X$-modules

$$C^{-1} : \Omega_X^i(\log E) \xrightarrow{\sim} \mathcal{H}^i(F_*\Omega_X^\bullet(\log E))$$

for $i = 0, 1, \ldots, d$.

Remark (2.2.1). It is usual to use the relative Frobenius morphism $F_{\mathrm{rel}} : X \to X' = X \times_k k^{1/p}$ to define the Cartier operator. In our situation the perfectness of the base field k allows us to use the absolute Frobenius F instead.

The following lemma is easily varified by local calculation.

Lemma (2.3). *Let the situation be as in* (2.1), *and* $B = \sum r_j E_j$ *be an effective integral divisor supported in* E *such that* $0 \leq r_j \leq p-1$ *for each* j. *Then we have a naturally induced complex* $\Omega_X^\bullet(\log E)(B) = \Omega_X^\bullet(\log E) \otimes \mathcal{O}_X(B)$ *of* $\mathcal{O}_X^p$*-modules, and the inclusion map*

$$\Omega_X^\bullet(\log E) \hookrightarrow \Omega_X^\bullet(\log E)(B)$$

is a quasi-isomorphism.

(2.4) In (2.3), if we denote the i-th cocycle and the i-th coboundary of the complex $F_*(\Omega_X^\bullet(\log E)(B))$ by $\mathcal{Z}^i$ and $\mathcal{B}^i$, respectively, then we have the exact sequences of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Z}^i \longrightarrow F_*(\Omega_X^i(\log E)(B)) \longrightarrow \mathcal{B}^{i+1} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{B}^i \longrightarrow \mathcal{Z}^i \longrightarrow \Omega_X^i(\log E) \longrightarrow 0$$

for $i = 0, 1, \ldots, d$. Here we note that the upper exact sequence for $i = 0$ is nothing but

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F^*} F_*(\mathcal{O}_X(B)) \longrightarrow \mathcal{B}^1 \longrightarrow 0.$$

3 Main results

Before stating the main theorem let us recall the following well-known

Definition (3.0). Let Y be a normal variety over a field of characteristic 0. A point $y \in Y$ is said to be a *rational singularity* if for a resolution of singularity $f : X \to Y$, one has $(R^i f_* \mathcal{O}_X)_y = 0$ for all $i > 0$. This property does not depend on the choice of a resolution.

Remark (3.0.1). The Grauert-Riemenschneider vanishing theorem [GR] in characteristic 0 guarantees that rational singularities are Cohen-Macaulay.

Theorem (3.1). *Let* R *be a finitely generated algebra over a field* k *of characteristic zero. If* R *has at most isolated rational singularities, then* R *is of* F*-rational type.*

Outline of Proof. We may assume that $Y = \mathrm{Spec} R$ has a unique singular point y. Let $f : X \to Y$ be a "good" resolution of singularity $y \in Y$, that is, a resolution whose exceptional set $E = f^{-1}(y)$ is a simple normal crossing divisor on X. One has an f-ample $\mathbf{Q}$-Cartier divisor D supported on E such that $[-D] = 0$.

Now we replace all the objects over k by objects over a finitely generated $\mathbf{Z}$-subalgebra A of k which give back the original ones after tensoring k over A, and look at closed fibers over $\mathrm{Spec} A$. Then all of the above mentioned properties are preserved for general closed fibers under the reduction process. So, from now on, we will use the same symbols $f : X \to Y = \mathrm{Spec} R$ etc., to denote their modulo p reductions, and assume that *everything is in characteristic* p.

Our goal is to show that R is F-rational for "$p \gg 0$" if it is Cohen-Macaulay of $\dim R = d$ and if $H^{d-1}(X, \mathcal{O}_X) = 0$. For this purpose we may replace R by its local ring $\mathcal{O}_{Y,y}$ at the unique singular point, and assume that (R, m) is local.

Next we observe the following, which follows from (2.4).

Proposition (3.2). *Let the situation be as above. Then the induced Frobenius map*

$$F : H^d_E(X, \mathcal{O}_X(-D)) \longrightarrow H^d_E(X, \mathcal{O}_X(-pD))$$

is injective if the following vanishing of cohomologies hold:

(a) $H^j_E(X, \Omega^i_X(\log E)(-D)) = 0$ *for* $i + j = d - 1$ *and* $i > 0$.

(b) $H^j_E(X, \Omega^i_X(\log E)(-pD)) = 0$ *for* $i + j = d$ *and* $i > 0$.

If $E \subset X$ admits a lifting to the ring of second Witt vectors and if $p > d$, then vanishing (a) holds true (cf. proof of [DI, Corollaire 2.11], together with (2.3)). However, as we are considering modulo p reduction from characteristic 0, there is a closed point $s \in S = \mathrm{Spec}A$ with $\mathrm{char}(\kappa(s)) > d$ such that the reduction to $\kappa(s)$ satisfies the lifting property, so that vanishing (a) holds for the fiber over every closed point in a open neighborhood of $s \in S$. Similarly does vanishing (b) for $p \gg 0$ by the Serre vanishing theorem.

Thus, (3.2) says that for "general" modulo p reduction, the e-times iterated Frobenius map

$$F^e : H^d_E(X, \mathcal{O}_X) \to H^d_E(X, \mathcal{O}_X(-p^e D))$$

is injective for every $e > 0$. For each $q = p^e$ we consider the commutative diagram

$$\begin{array}{ccccccc}
H^{d-1}(X, \mathcal{O}_X) = 0 & \to & H^d_m(R) & \to & H^d_E(X, \mathcal{O}_X) & \to 0 \\
\downarrow & & \downarrow F^e & & \downarrow F^e & \\
H^{d-1}(X, \mathcal{O}_X(-qD)) & \to & H^d_m(R) & \to & H^d_E(X, \mathcal{O}_X(-qD)) & \to 0
\end{array}$$

with exact rows. Let us define a decreasing filtration on $H^d_m(R)$ by

$$Filt^n(H^d_m(R)) := \mathrm{Image}(H^{d-1}(X, \mathcal{O}_X(-qD)) \to H^d_m(R)).$$

Then one can verify that $\bigcup_{n \in \mathbf{Z}} Filt^n(H^d_m(R)) = H^d_m(R)$ (cf. [TW]).

Now suppose that R is not F-rational. Then there exists a non-zero element $\xi \in (0)^*$ in $H^d_m(R)$, and $\xi^q := F^e(\xi) \notin Filt^{-q}(H^d_m(R))$ for all $q = p^e$ from the above diagram.

On the other hand, we can choose an integer $N > 0$ such that all non-zero elements of m^N are test elements (1.4). Since $(0 : m^N)$ in $H^d_m(R)$ is a finitely generated R-module, one has $(0 : m^N) \subseteq Filt^{n_0}(H^d_m(R))$ for some $n_0 \in \mathbf{Z}$.

Thus, if we pick a power $q = p^e \geq n_0$, then $\xi^q \notin (0 : m^N)$ in $H^d_m(R)$. Hence there is some test element $c \in m^N$ such that $c\xi^q \neq 0$. This contradicts $\xi \in (0)^*$, and we are done.

Example (3.3) [HW]. If R is a two-dimensional graded ring, then it is possible to know for what p the reduction modulo p is F-rational: Let R be a two-dimensional normal graded ring over a perfect field of characteristic $p > 0$. Such R can be represented by a smooth curve $X = \mathrm{Proj}R$ and a $\mathbf{Q}$-divisor D as

$$R = R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n.$$

Then R is F-rational if it is a rational singularity (i.e., $X = \mathbf{P}^1$ and $a(R) < 0$) and if $p \deg D > \deg D' - 2$, where D' is the "fractional part" of D. Even more, we can give a necessary and sufficient condition for R to be F-rational in terms of numerical data involving p and the coefficients of D.

Definition (3.4) (cf. [KMM]). Let Y be a normal variety over a field of characteristic 0. Y is said to have log terminal singularity if the following two conditions are satisfied:

(i) Y is $\mathbf{Q}$-Gorenstein, i.e., the canonical divisor K_Y of Y is $\mathbf{Q}$-Cartier.

(ii) Let $f : X \to Y$ be a good resolution of singularity. Condition (i) allows us to write

$$K_X = f^* K_Y + \sum_{i=1}^{r} a_i E_i$$

for some $a_i \in \mathbf{Q}$, where K_X is the canonical divisor of X and $E_1, \ldots, E_r$ are the irreducible components of the exceptional divisor of f. Then $a_i > 0$ for every i.

Remark (3.4.1). We have the similar implications as (1.2.1):

regular $\Rightarrow$ log terminal $\Rightarrow$ rational $\Rightarrow$ Cohen-Macaulay and normal.

In [W], Watanabe proved that a ring in characteristic 0 has log terminal singularity if it is of F-regular type and $\mathbf{Q}$-Gorenstein. Conversely we have

Theorem (3.5). *Let R be a finitely generated algebra over a field of characteristic zero. If R has at most isolated log terminal singlarities, then R is of F-regular type.*

Proof. We can easily reduce our statement to (3.1) using the canonical covering of R.

Example (3.6). Let X be a smooth del Pezzo surface (i.e., a smooth surface with ample anti-canonical divisor $-K_X$) of characteristic $p > 0$. Then $R = R(X, -K_X)$ has at most isolated log terminal singularity. In this case we can explicitly describe a condition for $R = R(X, -K_X)$ to be F-regular in terms of p and the self intersection number K_X^2. R is F-regular except for the following three cases:

(i) $K_X^2 = 3$ and $p = 2$.

(ii) $K_X^2 = 2$ and $p = 2$ or 3.

(iii) $K_X^2 = 1$ and $p = 2$, 3 or 5.

Moreover, there are both of F-regular and non F-regular cases for each of (i), (ii) and (iii). For example, in case (i) R is not F-regular if and only if X is isomorphic to the Fermat cubic surface in $\mathbf{P}^3$.

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