A characterization of rational singularities in terms of injectivity of Frobenius maps

(Frobenius写像の単射性による有理特異点の特徴付け)

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Introduction

In [HH1], Hochster and Huneke introduced the notion of the tight closure of an ideal in a ring of characteristic \( p > 0 \). Tight closure enables us to define classes of rings of characteristic \( p \) such as \( F \)-rational rings [FW] and \( F \)-regular rings [HH1], and it turns out that they are closely related with some classes of singularities in characteristic 0 defined via resolution of singularity.

It was shown by Smith [S] that a ring in characteristic 0 has a rational singularity if its modulo \( p \) reduction is \( F \)-rational for infinitely many \( p \), and Watanabe [W] obtained an analogous result for log terminal singularity and \( F \)-regularity. The essential parts of these results hold true in arbitrary positive characteristic. But if we consider the converse implication, we soon confront some difficulty arising from pathological phenomena in small characteristic \( p > 0 \). For example, a two-dimensional log terminal singularity is always \( F \)-regular for \( p > 5 \), but is not \( F \)-regular in general if \( p = 2, 3 \) or 5 [Ha]. To avoid such difficulty we will look at generic behavior of modulo \( p \) reduction for sufficiently large \( p \).

In this context, Fedder gave affirmative answer in some special cases [F1,2]: If a graded ring \( R \) over a field of characteristic 0 has a rational singularity and if \( R \) is a complete intersection or \( \dim R = 2 \), then modulo \( p \) reduction of \( R \) is \( F \)-rational for \( p \gg 0 \).

Unfortunately, except for the above special cases, the implication "rational singularity \( \Rightarrow F \)-rational" remained to be open, and is considered to be one of the fundamental problems in the tight closure theory. We aim to give an affirmative answer to this question in a fairly general situation.

A \( d \)-dimensional Cohen-Macaulay local ring \((R, m)\) of characteristic \( p > 0 \) is \( F \)-rational if and only if the tight closure \((0)^*\) of \((0)\) in \( H^d_d(R) \) coincides \((0)\) itself. Let us assume that \( R \) has an isolated singularity and that there is a "good" resolution of singularity \( f : X \to Y = \text{Spec} R \), that is, a resolution with simple normal crossing exceptional divisor \( E \) (in characteristic 0, such a resolution exists [Hi]). If \( D \) is an \( f \)-ample fractional divisor such that \( -D \) has no integral part, then we observe that \( R \) is \( F \)-rational if it has at most "rational" singularity (i.e., \( H^i(X, \mathcal{O}_X) = 0 \) for \( i > 0 \)) and if the iterated Frobenius map

\[
F^e : H^d_d(X, \mathcal{O}_X) \to H^d_d(X, \mathcal{O}_X(-qD))
\]

is injective for all powers \( q = p^e \). This is a generalization of Fedder and Watanabe's result for graded rings [FW].
To analyze the above Frobenius maps we will use log de Rham complex \( \Omega^*_A (\log E) \) and the Cartier operator \([C], [Ka]\). We see that an obstruction for the map to be injective lies in non-vanishing of certain cohomology groups (3.2). However, a slight generalization of Deligne and Illusie's proof of the Akizuki-Kodaira-Nakano vanishing theorem for characteristic \( p > 0 \) [DI] and the Serre vanishing theorem imply that these cohomology groups vanish for \( p > 0 \) if we reduce \( X \) and \( D \) from characteristic 0 to characteristic \( p > 0 \).

Consequently, our argument establishes the correspondence of rational singularity with \( F \)-rationality (3.1), and also that of log terminal singularity with \( F \)-regularity (3.5).

\section{Preliminaries}

Let \( R \) denote a Noetherian ring. We will often assume that \( R \) has prime characteristic \( p > 0 \). In this case we always use the letter \( q \) for a power \( p^e \) of \( p \). Also, \( R^0 \) will denote the set of elements of \( R \) which is not in any minimal prime ideal.

**Definition (1.1) [HH1].** Let \( R \) be a Noetherian ring of characteristic \( p > 0 \), and \( I \subset R \) be an ideal. The tight closure \( I^* \) of \( I \) in \( R \) is the ideal defined by \( x \in I^* \) iff there exists \( c \in R^0 \) such that \( cx^q \in I^{[q]} \) for \( q = p^e \gg 0 \), where \( I^{[q]} \) is the ideal generated by the \( q \)-th powers of the elements of \( I \). We say that \( I \) is tightly closed if \( I^* = I \).

**Definition (1.2).** Let \( R \) denote a Noetherian ring of characteristic \( p > 0 \).

(i) [FW] A local ring \((R, m)\) is said to be \( F \)-rational if some (or, equivalently, every) ideal generated by system of parameters of \( R \) is tightly closed. When \( R \) is not local, we say that \( R \) is \( F \)-rational if every localization is \( F \)-rational.

(ii) [HH1] \( R \) is said to be \( F \)-regular if every ideal of \( R \) is tightly closed.

**Remark (1.2.1).** In characteristic \( p > 0 \), the following implications are known [HH1]:

\[ \text{regular} \Rightarrow \text{\( F \)-regular} \Rightarrow \text{\( F \)-rational} \Rightarrow \text{normal}. \]

Also, an \( F \)-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay, and a Gorenstein \( F \)-rational ring is \( F \)-regular.

**Definition (1.3).** Let \( R \) be a Noetherian ring of characteristic \( p > 0 \). An element \( c \in R^0 \) is said to be a test element if for all ideals \( I \subset R \) and \( x \in R \), one has

\[ x \in I^* \iff cx^q \in I^{[q]} \text{ for all } q = p^e \ (e \geq 0). \]

**Proposition (1.4) [HH2].** If \( R \) is a reduced excellent local ring of characteristic \( p > 0 \), and \( c \in R^0 \) is an element such that \( R_c \) is regular, then some power of \( c \) is a test element for \( R \).

Given a property \( P \) defined for rings of characteristic \( p > 0 \) such as "\( F \)-rational" or "\( F \)-regular", we will extend the concept to characteristic 0 using the technique of reduction modulo \( p \).

**Definition (1.5) (cf. [HR]).** Let \( R \) be a finitely generated algebra over a field \( k \) of characteristic 0. We say that \( R \) is of \( P \) type if there exist a finitely generated \( \mathbb{Z} \)-subalgebra \( A \) of \( k \) and a finitely generated \( A \)-algebra \( R_A \) satisfying the following conditions:

(i) \( R_A \) is flat over \( A \) and \( R_A \otimes_A k \cong R \).
(ii) $R_\kappa = R_A \otimes_A \kappa(s)$ has property $P$ for every closed point $s$ in a dense open subset of $S = \text{Spec}A$, where $\kappa = \kappa(s)$ is the residue field of $s \in S$.

**Remark** (1.5.1). In condition (ii), the fiber ring $R_\kappa = R_A \otimes_A \kappa$ always has positive characteristic since $A$ is finitely generated over $\mathbb{Z}$. We sometimes abbreviate the statement in condition (ii) as "$R_\kappa$ has property $P$ for general closed points $s \in S$ with residue field $\kappa = \kappa(s)$". However, if $R$ is of $P$ type, we can replace $S = \text{Spec}A$ by a suitable open subset so that condition (ii) holds for every closed point $s \in S$.

A normal ring $R$ in characteristic 0 is said to have rational singularity if for a resolution of singularity $f : X \to \text{Spec}R$, one has $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. The aim of the present paper is to show the converse of the following result due to Smith [S].

**Theorem** (1.6) [S]. Let $R$ be a finitely generated algebra over a field of characteristic zero. If $R$ is of $F$-rational type, then it has at most rational singularity.

### 2 Log de Rham complex and the Cartier operator

We will review some fundamental facts about log de Rham complex and the Cartier operator in characteristic $p > 0$. Concerning these subjects the reader may consult [C] and [Ka] (see also [EV]).

**Assumption** (2.1). Throughout this section $X$ will denote a $d$-dimensional smooth variety of finite type over a perfect field $k$ of characteristic $p > 0$, and $E = \sum_{j=1}^n E_j$ a reduced simple normal crossing divisor on $X$, that is, a divisor with smooth irreducible components $E_j$ intersecting transversally.

Let us choose local parameters $t_1, \ldots, t_d$ of $X$ so that $E$ is locally defined by $t_1 \cdots t_s = 0$. Then we can consider the locally free $\mathcal{O}_X$-module $\Omega^i_X(\log E)$ with local basis

$$\frac{dt_1}{t_1}, \ldots, \frac{dt_s}{t_s}, dt_{s+1}, \ldots, dt_d.$$  

We define $\Omega^i_X(\log E) = \bigwedge^i \Omega^1_X(\log E)$ for $i \geq 0$. These sheaves, together with the differential maps $d$, give rise to a complex $\Omega^\bullet_X(\log E)$ called a log de Rham complex.

(2.2) **The Cartier operator** [C], [Ka]. Let $F : X \to X$ be the absolute Frobenius morphism of $X$. The direct image $F_* \Omega^i_X(\log E)$ of the de Rham complex can be viewed as a complex of $\mathcal{O}_X$-modules via $F^* : \mathcal{O}_X \to F_* \mathcal{O}_X$. We denote the $i$-th cohomology sheaf of this complex by $\mathcal{H}^i(F_* \Omega^i_X(\log E))$. Then, there is an isomorphism of $\mathcal{O}_X$-modules

$$C^{-1} : \Omega^i_X(\log E) \xrightarrow{\sim} \mathcal{H}^i(F_* \Omega^i_X(\log E))$$  

for $i = 0, 1, \ldots, d$.

**Remark** (2.2.1). It is usual to use the relative Frobenius morphism $F_{\text{rel}} : X \to X' = X \times_k k^{1/p}$ to define the Cartier operator. In our situation the perfectness of the base field $k$ allows us to use the absolute Frobenius $F$ instead.

The following lemma is easily verified by local calculation.
Lemma (2.3). Let the situation be as in (2.1), and $B = \sum r_jE_j$ be an effective integral divisor supported in $E$ such that $0 \leq r_j \leq p - 1$ for each $j$. Then we have a naturally induced complex $\Omega^*_X(\log E)(B) = \Omega^*_X(\log E) \otimes \mathcal{O}_X(B)$ of $\mathcal{O}_X^*$-modules, and the inclusion map

$$\Omega^*_X(\log E) \hookrightarrow \Omega^*_X(\log E)(B)$$

is a quasi-isomorphism.

(2.4) In (2.3), if we denote the $i$-th cocycle and the $i$-th coboundary of the complex $F_*\omega_X(\log E)(B)$ by $\mathcal{Z}^i$ and $\mathcal{B}^i$, respectively, then we have the exact sequences of $\mathcal{O}_X$-modules

$$0 \rightarrow \mathcal{Z}^i \rightarrow F_*\omega_X(\log E)(B) \rightarrow \mathcal{B}^{i+1} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{B}^i \rightarrow \mathcal{Z}^i \rightarrow \Omega^i_X(\log E) \rightarrow 0$$

for $i = 0, 1, \ldots, d$. Here we note that the upper exact sequence for $i = 0$ is nothing but

$$0 \rightarrow \mathcal{O}_X \xrightarrow{F^*} F_*\omega_X(\mathcal{O}_X(B)) \rightarrow \mathcal{B}^1 \rightarrow 0.$$

3 Main results

Before stating the main theorem let us recall the following well-known

Definition (3.0). Let $Y$ be a normal variety over a field of characteristic 0. A point $y \in Y$ is said to be a rational singularity if for a resolution of singularity $f : X \rightarrow Y$, one has $(R^1f_*\mathcal{O}_X)_y = 0$ for all $i > 0$. This property does not depend on the choice of a resolution.

Remark (3.0.1). The Grauert-Riemenschneider vanishing theorem [GR] in characteristic 0 guarantees that rational singularities are Cohen-Macaulay.

Theorem (3.1). Let $R$ be a finitely generated algebra over a field $k$ of characteristic zero. If $R$ has at most isolated rational singularities, then $R$ is of $F$-rational type.

Outline of Proof. We may assume that $Y = \text{Spec}R$ has a unique singular point $y$. Let $f : X \rightarrow Y$ be a “good” resolution of singularity $y \in Y$, that is, a resolution whose exceptional set $E = f^{-1}(y)$ is a simple normal crossing divisor on $X$. One has an $f$-ample $\mathbb{Q}$-Cartier divisor $D$ supported on $E$ such that $[-D] = 0$.

Now we replace all the objects over $k$ by objects over a finitely generated $\mathbb{Z}$-subalgebra $A$ of $k$ which give back the original ones after tensoring $k$ over $A$, and look at closed fibers over $\text{Spec}A$. Then all of the above mentioned properties are preserved for general closed fibers under the reduction process. So, from now on, we will use the same symbols $f : X \rightarrow Y = \text{Spec}R$ etc., to denote their modulo $p$ reductions, and assume that everything is in characteristic $p$.

Our goal is to show that $R$ is $F$-rational for “$p \gg 0$” if it is Cohen-Macaulay of $\dim R = d$ and if $H^{d-1}(X, \mathcal{O}_X) = 0$. For this purpose we may replace $R$ by its local ring $\mathcal{O}_{Y,y}$ at the unique singular point, and assume that $(R, m)$ is local.
Next we observe the following, which follows from (2.4).

**Proposition (3.2).** Let the situation be as above. Then the induced Frobenius map

\[ F : H^d_E(X, \mathcal{O}_X(-D)) \rightarrow H^d_E(X, \mathcal{O}_X(-pD)) \]

is injective if the following vanishing of cohomologies hold:

(a) \( H^i_E(X, \Omega^j_X(\log E)(-D)) = 0 \) for \( i + j = d - 1 \) and \( i > 0 \).

(b) \( H^i_E(X, \Omega^j_X(\log E)(-pD)) = 0 \) for \( i + j = d \) and \( i > 0 \).

If \( E \subset X \) admits a lifting to the ring of second Witt vectors and if \( p > d \), then vanishing (a) holds true (cf. proof of [DI, Corollaire 2.11], together with (2.3)). However, as we are considering modulo \( p \) reduction from characteristic 0, there is a closed point \( s \in S = \text{Spec} A \) with \( \text{char}(\kappa(s)) > d \) such that the reduction to \( \kappa(s) \) satisfies the lifting property, so that vanishing (a) holds for the fiber over every closed point in an open neighborhood of \( s \in S \). Similarly does vanishing (b) for \( p \gg 0 \) by the Serre vanishing theorem.

Thus, (3.2) says that for “general” modulo \( p \) reduction, the \( e \)-times iterated Frobenius map

\[ F^e : H^d_E(X, \mathcal{O}_X) \rightarrow H^d_E(X, \mathcal{O}_X(-p^eD)) \]

is injective for every \( e > 0 \). For each \( q = p^e \) we consider the commutative diagram

\[
\begin{array}{ccc}
H^{d-1}(X, \mathcal{O}_X) = 0 & \rightarrow & H^d_m(R) \rightarrow H^d_E(X, \mathcal{O}_X) \rightarrow 0 \\
\downarrow & & \downarrow F^e \\
H^{d-1}(X, \mathcal{O}_X(-qD)) & \rightarrow & H^d_m(R) \rightarrow H^d_E(X, \mathcal{O}_X(-qD)) \rightarrow 0
\end{array}
\]

with exact rows. Let us define a decreasing filtration on \( H^d_m(R) \) by

\[ \text{Filt}^n(H^d_m(R)) := \text{Image}(H^{d-1}(X, \mathcal{O}_X(-qD)) \rightarrow H^d_m(R)). \]

Then one can verify that \( \bigcup_{n \in \mathbb{Z}} \text{Filt}^n(H^d_m(R)) = H^d_m(R) \) (cf. [TW]).

Now suppose that \( R \) is not \( F \)-rational. Then there exists a non-zero element \( \xi \in \langle 0 \rangle^* \) in \( H^d_m(R) \), and \( \xi^q := F^e(\xi) \notin \text{Filt}^{-q}(H^d_m(R)) \) for all \( q = p^e \) from the above diagram.

On the other hand, we can choose an integer \( N > 0 \) such that all non-zero elements of \( m^N \) are test elements (1.4). Since \( (0 : m^N) \) in \( H^d_m(R) \) is a finitely generated \( R \)-module, one has \( (0 : m^N) \subseteq \text{Filt}^n_0(H^d_m(R)) \) for some \( n_0 \in \mathbb{Z} \).

Thus, if we pick a power \( q = p^e \geq n_0 \), then \( \xi^q \notin (0 : m^N) \) in \( H^d_m(R) \). Hence there is some test element \( c \in m^N \) such that \( c\xi^q \neq 0 \). This contradicts \( \xi \in \langle 0 \rangle^* \), and we are done.

**Example (3.3) [HW].** If \( R \) is a two-dimensional graded ring, then it is possible to know for what \( p \) the reduction modulo \( p \) is \( F \)-rational: Let \( R \) be a two-dimensional normal graded ring over a perfect field of characteristic \( p > 0 \). Such \( R \) can be represented by a smooth curve \( X = \text{Proj} R \) and a \( \mathbb{Q} \)-divisor \( D \) as

\[ R = R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n. \]

Then \( R \) is \( F \)-rational if it is a rational singularity (i.e., \( X = \mathbb{P}^1 \) and \( a(R) < 0 \)) and if \( p \deg D > \deg D' - 2 \), where \( D' \) is the “fractional part” of \( D \). Even more, we can give
a necessary and sufficient condition for $R$ to be $F$-rational in terms of numerical data involving $p$ and the coefficients of $D$.

**Definition (3.4)** (cf. [KMM]). Let $Y$ be a normal variety over a field of characteristic 0. $Y$ is said to have log terminal singularity if the following two conditions are satisfied:

(i) $Y$ is $\mathbb{Q}$-Gorenstein, i.e., the canonical divisor $K_Y$ of $Y$ is $\mathbb{Q}$-Cartier.

(ii) Let $f : X \to Y$ be a good resolution of singularity. Condition (i) allows us to write

$$K_X = f^*K_Y + \sum_{i=1}^{r} a_i E_i$$

for some $a_i \in \mathbb{Q}$, where $K_X$ is the canonical divisor of $X$ and $E_1, \ldots, E_r$ are the irreducible components of the exceptional divisor of $f$. Then $a_i > 0$ for every $i$.

**Remark (3.4.1).** We have the similar implications as (1.2.1):

regular $\Rightarrow$ log terminal $\Rightarrow$ rational $\Rightarrow$ Cohen-Macaulay and normal.

In [W], Watanabe proved that a ring in characteristic 0 has log terminal singularity if it is of $F$-regular type and $\mathbb{Q}$-Gorenstein. Conversely we have

**Theorem (3.5).** Let $R$ be a finitely generated algebra over a field of characteristic zero. If $R$ has at most isolated log terminal singularities, then $R$ is of $F$-regular type.

**Proof.** We can easily reduce our statement to (3.1) using the canonical covering of $R$.

**Example (3.6).** Let $X$ be a smooth del Pezzo surface (i.e., a smooth surface with ample anti-canonical divisor $-K_X$) of characteristic $p > 0$. Then $R = R(X, -K_X)$ has at most isolated log terminal singularity. In this case we can explicitly describe a condition for $R = R(X, -K_X)$ to be $F$-regular in terms of $p$ and the self intersection number $K_X^2$. $R$ is $F$-regular except for the following three cases:

(i) $K_X^2 = 3$ and $p = 2$.

(ii) $K_X^2 = 2$ and $p = 2$ or 3.

(iii) $K_X^2 = 1$ and $p = 2, 3$ or 5.

Moreover, there are both of $F$-regular and non $F$-regular cases for each of (i), (ii) and (iii). For example, in case (i) $R$ is not $F$-regular if and only if $X$ is isomorphic to the Fermat cubic surface in $\mathbb{P}^3$.

**References**


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