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A generalization of Kohnen's estimates for Fourier coefficients of Siegel cusp forms

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The purpose of this article is to show that the main result of [K] is valid for any level.

Theorem. *Let F be a cusp form of integral or half integral weight $k(>2)$ with respect to the subgroup $\Gamma_2(N)$ of $\mathrm{Sp}_2(\mathbf{Z})$, where*

$$\Gamma_2(N):=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_2 \middle|\, C\equiv 0 \pmod N\right\}.$$

And let its Fourier expansion be given by

$$F(Z)=\sum_T a(T)\exp(2\pi i\,\mathrm{tr}\,T\langle Z\rangle),$$

where T runs over positive definite symmetric half-integral 2×2-matrices. Then we have

$$a(T)\ll_{\varepsilon,F}(\min T)^{5/18+\varepsilon}(\det T)^{(k-1)/2+\varepsilon}\quad(\forall\varepsilon>0),\tag{1}$$

where min$T$ *is the smallest positive integer represented by T.*

The idea to prove Theorem is the same as in [K], that is a combination of appropriate estimates for both Fourier coefficients of Jacobi Poincaré series and Petterson norms of Fourier-Jacobi coefficients of Siegel modular forms.

$\mathcal{H}_i$ denotes the Siegel upper half space of degree i consisting of complex $i\times i$-matrices with positive definite imaginary part. We often write

$$Z=X+iY=\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}=:\begin{pmatrix}u+iv & x+iy\\ x+iy & u'+iv'\end{pmatrix}\in\mathcal{H}_2.$$

For simplicity, we condider only the integral weight case.

Proposition 1. *We let $\Gamma_1^J(N)$ be the Jacobi group which is the semi direct product of $\Gamma_1(N)$ and $\mathbf{Z}^2$, and let $J_{k,m}^{cusp}(N)$ be the space of holomorphic Jacobi cusp forms on $\mathcal{H}_1\times\mathbf{C}$ of weight k and index m with respect to $\Gamma_1^J(N)$ (cf. e.g. [E-Z]).*

For ϕ in $J_{k,m}^{cusp}(N)$, let $c(n,r)$ be the (n,r)-th Fourier coefficient of ϕ $(n,r \in \mathbf{Z},\ r^2 < 4mn)$. Put $D = r^2 - 4mn$. Then we have

$$c(n,r) \ll_{\varepsilon,k} (m + |D|^{1/2+\varepsilon})^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} ||\phi|| \quad (\forall \varepsilon > 0)$$

where the constant implied in $\ll$ depends only on ε and k (not on m).

Proof. Let $P_{n,r} = P_{k,m,n,r}$ be the (n,r)-th Jacobi Poincaré series in $J_{k,m}^{cusp}(N)$ characterized by

$$\langle \psi, P_{n,r} \rangle = \lambda_{k,m,D} b_{n,r}(\psi) \quad (\forall \psi \in J_{k,m}^{cusp}(N))$$

where $b_{n,r}(\psi)$ denotes the (n,r)-th Fourier coefficients of ψ and

$$\lambda_{k,m,D} := \frac{1}{2}\Gamma\left(k - \frac{3}{2}\right) \pi^{-k+3/2} m^{k-2} |D|^{-k+3/2}.$$

Then the Cauchy-Schwarz inequality gives

$$|c(n,r)|^2 \le \lambda_{k,m,D}^{-2} ||\phi||^2 \langle P_{n,r}, P_{n,r} \rangle = \lambda_{k,m,D}^{-1} b_{n,r}(P_{n,r}) ||\phi||^2.$$

We can show that the Fourier coefficient of $P_{n,r}$ as follows (cf. [G-K-Z], p.519);

$$b_{n,r}(P_{n,r}) = 1 + (-1)^k \delta_m(r) + \frac{i^k \pi \sqrt{2}}{\sqrt{m}} \sum_{N|c \ge 1} c^{-3/2} (\exp(r^2/2mc) H_{m,c}^{+}(n,r)$$
$$+ (-1)^k \exp(-r^2/2mc) H_{m,c}^{-}(n,r)) J_{k-3/2}\left(\frac{\pi |D|}{mc}\right),$$

where

$$\delta_m(r) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases},$$

$J_{k-3/2}$ is the modified Bessel function of order $k - 3/2$, and

$$H_{m,c}^{\pm}(n,r) := \sum_{x(c),y(c)^*} \mathbf{e}_c((mx^2 + rx + n)\bar{y} + ny \pm rx),$$

where x resp. y run through $\mathbf{Z}/c\mathbf{Z}$ resp. $(\mathbf{Z}/c\mathbf{Z})^*$, $\bar{y}$ denotes an inverse of $y \pmod{c}$, $\mathbf{e}_c(b) := \exp(2\pi i b/c)$ for $c \in \mathbf{N}$, $b \in \mathbf{Z}/c\mathbf{Z}$, $\varepsilon(y) = 1$ or i according as $y \equiv 1 \pmod{4}$ or $\equiv 3 \pmod{4}$, and $\left(\frac{*}{*}\right)$ means the Kronecker symbol. $H_{m,c}^{\pm}(n,r)$ is a certain character sum, which is Gauss sum for x and Kloosterman sum for y, and by factorizing c to prime powers, for $D := r^2 - 4mn$ we can prove an estimate

$$H_{m,c}^{\pm}(n,r) \ll_{\varepsilon} c^{1+\varepsilon}(D,c) \quad (\forall \varepsilon > 0).$$

From this and the estimate

$$J_{k-3/2}(x) \ll_k \min\{x^{-1/2},\ x^{k-3/2}\} \quad (x > 0)$$

(cf. e.g. [B], p.4 and p.74), we easily find

$$b_{n,r}(P_{n,r}) \ll_{\varepsilon,k} 1 + \frac{|D|^{1/2+2\varepsilon}}{m}$$

for any $\varepsilon > 0$ and complete the proof.

□

To estimate Petterson norm $||\phi||$, for an analogue of the Rankin convolution series

$$D_{F,F}(s) := \zeta(2s - 2k + 4) \sum_{n \geq 1} \langle \phi_n, \phi_n \rangle n^{-s}$$

where

$$F(Z) = \sum_{n \geq 1} \phi_n(\tau, z) \exp(2\pi i n \tau'),$$

we want to use the following Landau's Theorem;

Theorem (Landau-Shintani). *Suppose that*

$$\xi(s) = \sum_{n \geq 1} c(n)\, n^{-s},\ \xi_i(s) = \sum_{n \geq 1} c_i(n)\, n^{-s}\ (1 \leq i \leq I)$$

are Dirichlet serieses with non-negative cefficients which converge for $Re(s) > \sigma_0$, have meromorphic continuation to $\mathbf{C}$ *with finitely many poles and satisfy a functional equation*

$$\xi^*(\delta - s) = \sum_{i=1}^{I} \xi_i^*(s)$$

where

$$\xi_i^*(s) = B\, A^s \prod_{j=1}^{J} \Gamma(a_j s + b_j)\, \xi(s) \quad (A \in \mathbf{C},\ B \in \mathbf{C},\ a_j > 0,\ b_j \in \mathbf{R}),$$

$$\xi_i^*(s) = B_i\, A_i^s \prod_{j=1}^{J} \Gamma(a_j s + b_j)\, \xi(s) \quad (A_i \in \mathbf{C},\ B_i \in \mathbf{C},\ a_j \text{ and } b_j \text{ are same as above}).$$

Suppose

$$\kappa := (2\sigma_0 - \delta) \sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$

Then we have

$$\sum_{n \leq x} c(n) = \sum_{s:\text{all poles}} \operatorname{Res}\left(\frac{\xi(s)}{s} x^s \right) + O_\eta(x^\eta)$$

for any $\eta > \eta_0 := \{\delta + \sigma_0(\kappa - 1)\}/(\kappa + 1)$.

For the proof, see Theorem 3 and its proof in [S-S].

□

The central extension of $\Gamma_1^J(N)$ by $\mathbf{Z}$ is embedded into $\Gamma_2(N)$ via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu, \kappa \right) \mapsto \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\lambda,\ \mu) = (\lambda',\ \mu') \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and we denote by C_N the image in $\Gamma_2(N)$. Denote the left upper entry of $Z \in \mathcal{H}_2$ by Z_1. For a natural number N, $Z \in \mathcal{H}_2$ and $s \in \mathbf{C}$ with $Re(s) \gg 2$ we define a Klingen-Siegel type Eisenstein series

$$E_{s,N}(Z) := \sum_{M \in C_N \backslash \Gamma_2(N)} \left(\frac{\det \operatorname{Im} M\langle Z \rangle}{\operatorname{Im} M\langle Z \rangle_1} \right)^s .$$

It is easily seen that this series is well defined, absolutery convergent, and invariant under the action of $\Gamma_2(N)$. We put

$$E^*_{s,N}(Z) := \pi^{-s}\, \Gamma(s)\, \zeta(2s)\, E_{s,N}(Z).$$

By Main Lemma on p.545 in [K-S], we know $E_{s,1}(Z)$ has a meromorphic continuation to $\mathbf{C}$, has only two poles at $s = 0, 2$ which are simple, and satisfies a functional equation

$$E^*_{2-s,1}(Z) = E^*_{s,1}(Z).$$

By the method of Rankin-Selberg convolution

$$\pi^{-k+2} \langle F E^*_{s-k+2,N}, F \rangle = D^*_{F,F}(s) \tag{2}$$

can be proved, and analytic properties of $D^*_{F,F}(s)$ follow from those of $E^*_{s,N}(s)$. But the functional equations are complicated.

The idea to prove Theorem for any level N is to write the functional equations satisfied by Eisenstein series as a form

$$E^*_{2-s,N}(Z) = \text{a linear combination of } E^*_{s,m}(Z)$$

where m is a natural number with $m|N$. This is necessary to apply Rankin's method.

Lemma 1. *$E_{s,N}(Z)$ has a meromorphic continuation to $\mathbf{C}$. Its poles are $s = 0$ and 2, which are simple. And it satisfies a functional equation*

$$E^*_{2-s,N}(Z) = \text{a finite sum of } \frac{\pm n^s}{P(s)} E^*_{s,m}(Z),$$

where m, n are natural numbers with $m|N$ and $P(s)$ is a finite product of $1 - \tilde{m}^{2(2-s)}$ with $\tilde{m}|m$.

Proof. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \in \Gamma_2(N)$, we notice that

$$\frac{\det \operatorname{Im} M\langle Z \rangle}{\operatorname{Im} M\langle Z \rangle_1} = \frac{|Y|}{Y\left[Z^* \begin{pmatrix} c_4 \\ -c_3 \end{pmatrix} + \begin{pmatrix} d_4 \\ -d_3 \end{pmatrix} \right]}$$

($Y \begin{bmatrix} a \\ b \end{bmatrix} := (\bar{a}, \bar{b})\, Y \begin{pmatrix} a \\ b \end{pmatrix}$, Z^* means the adjoint matrix of Z) and the mapping

$$\begin{pmatrix} * & * & * & * \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \mapsto (c_3, c_4, d_3, d_4)$$

induces a bijection between

$$C_N\backslash\Gamma_2(N) \text{ and } \{(c_3, c_4, d_3, d_4) \in \mathbf{Z}^4 | \text{primitive and } c_3 \equiv c_4 \equiv 0 \pmod{N}\}.$$

In the following sums, $c = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$, $d = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}$ run over $\mathbf{Z}^2$ under the condition that c_3, c_4, d_3, d_4 are relatively prime. In general, for a square free integer m and a natural number $l = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r} \in \mathbf{N}$ (where $p_1, p_2, \ldots, p_r$ are different prime numbers and $e_i > 0$) it holds

$$
\begin{aligned}
& \frac{1}{l^s} E_{s,m}(lZ) \\
= & \sum_{\substack{({}^tc,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ m)}} \frac{|Y|^s}{(Y[Z^*lc+d])^s} \\
= & \left(\sum_{\substack{({}^tc,{}^td)=1 \\ (l,{}^td)\neq 1 \\ c\equiv 0(\mathrm{mod}\ m)}} + \sum_{\substack{({}^tc,{}^td)=1 \\ (l,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ m)}} \right) \frac{|Y|^s}{(Y[Z^*lc+d])^s} \\
= & \sum_{\substack{({}^tc,{}^td)=1 \\ d\equiv 0(\mathrm{mod}\ \exists p_i) \\ c\equiv 0(\mathrm{mod}\ m)}} \frac{|Y|^s}{(Y[Z^*lc+d])^s} + \sum_{\substack{({}^tc,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ lm)}} \frac{|Y|^s}{(Y[Z^*c+d])^s} \\
= & \sum_i \frac{1}{p_i^{2s}} \sum_{\substack{({}^tc,p_i^td)=1 \\ c\equiv 0(\mathrm{mod}\ m)}} \frac{|Y|^s}{(Y[Z^*(l/p_i)c+d])^s} \\
& - \sum_{i\neq j} \frac{1}{(p_ip_j)^{2s}} \sum_{\substack{({}^tc,p_ip_j^td)=1 \\ c\equiv 0(\mathrm{mod}\ m)}} \frac{|Y|^s}{(Y[Z^*(l/p_ip_j)c+d])^s} \\
& + \ldots \\
& + \sum_{\substack{({}^tc,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ lm)}} \frac{|Y|^s}{(Y[Z^*c+d])^s} \\
= & \sum_i \frac{1}{p_i^{2s}} \left(\sum_{\substack{({}^tc,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ 0)}} - \sum_{\substack{({}^tc,{}^td)=1 \\ c\equiv 0(\mathrm{mod}\ m) \\ c\equiv 0(\mathrm{mod}\ p_i)}} \right) \frac{|Y|^s}{(Y[Z^*(l/p_i)c+d])^s} \\
& - \ldots \\
= & \sum_i \frac{1}{(p_i)^{2s}} \{E_{s,m}((l/p_i)Z) - E_{s,\,\mathrm{l.c.m.}(m,p_i)}((l/p_i)Z)\} \\
& - \sum_{i\neq j} \frac{1}{(lp_ip_j)^s} \{E_{s,m}((l/p_ip_j)Z) - E_{s,\,\mathrm{l.c.m.}(m,p_i)}((l/p_ip_j)Z) - E_{s,\,\mathrm{l.c.m.}(m,p_j)}((l/p_ip_j)Z) \\
& \qquad + E_{s,\,\mathrm{l.c.m.}(m,p_ip_j)}((l/p_ip_j)Z)\} \\
& + \ldots
\end{aligned}
$$

$$+(-1)^{r-1}\frac{1}{(lp_1p_2\dots p_r)^s}\{E_{s,m}((l/p_1\dots p_r)Z)-\sum_i E_{s,\,\mathrm{l.c.m.}(m,p_i)}((l/p_1\dots p_r)Z)$$
$$+\sum_{i\neq j}E_{s,\,\mathrm{l.c.m.}(m,p_ip_j)}((l/p_1\dots p_r)Z)-\dots+(-1)^rE_{s,\,\mathrm{l.c.m.}(m,p_1\dots p_r)}((l/p_1\dots p_r)Z)\}$$
$$+E_{s,lm}(Z). \tag{3}$$

We apply (3) for $m=1$ and $l=N$; if N is not square-free the last term is $E_{s,N}(Z)$, otherwise the last two terms are $(-N^{-2s}+1)E_{s,N}(Z)$, and in the both cases the rests are $\pm\tilde{n}^{-s}E_{s,\tilde{m}}(\tilde{l}Z)$ where $\tilde{l},\tilde{m},\tilde{n}$ are natural numbers with $\tilde{l}\tilde{m}|N$, $\tilde{m}<N$. Hence for a non-square-free number N we have

$$E^*_{s,N}(Z)=\text{a finite sum of }\pm n^s\,E^*_{s,m}(lZ)$$

where l,m,n are natural numbers with $lm|N$, $m<N$, and for a square-free number N we have

$$(1-N^{2s})E^*_{s,N}(Z)=\text{a finite sum of the same type as above.}$$

So, by induction on N we deduce that $E_{s,N}(Z)$ has a meromorphic continuation to $\mathbf{C}$, has poles only at $s=0,2$ and satisfies a functional equation

$$E^*_{2-s,N}(Z)=\text{a finite sum of }\frac{\pm n^s}{P_1(s)}E^*_{s,m}(lZ)$$

where l,m,n are natural numbers with $lm|N$ and $P_1(s)$ is a finite product of $1-\tilde{m}^{2(2-s)}$ with $\tilde{m}|m$. Now we notice that (3) makes l smaller, and apply (3) repeatedly in all terms in this right-hand side until l becames 1, then finally we get the functional equation in Lemma 1.

□

Then we can use Rankin's method and deduce

Lemma 2. *Let the notations be as above, and take a natural number m with $m|N$. For $L\in\Gamma_2$, we write the Fourier expantions of $F(L^{-1}\langle Z\rangle)$ as*

$$F(L^{-1}\langle Z\rangle)=\sum_{n\geq1}\phi_{n,L}(\tau,z)\exp\left(\frac{2\pi i n\tau'}{N}\right).$$

We define a Dirichlet series $D_{F,F,m}(s)$ as $\zeta(2s-2k+4)$ times

$$\sum_{n\geq1}\left\{\sum_{L\in\Gamma_2(N)\backslash\Gamma_2(m)}\int_{\mathcal{F}}|\phi_{n,L}(\tau,z)|^2\exp\left(-\frac{4\pi n y^2}{vN}\right)v^{k-3}\,du\,dv\,dx\,dy\right\}n^{-s}$$

where $\mathcal{F}$ is a fundamental domain $\Gamma_1^J(m)\backslash\mathcal{H}_1\times\mathbf{C}$ (so $D_{F,F,N}(s)=D_{F,F}(s)$), and put

$$D^*_{F,F,m}(s):=(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-k+2)\,D_{F,F,m}(s).$$

Then we have

$$\pi^{-k+2}\langle FE^*_{s-k+2,m},F\rangle=N^s\,D^*_{F,F,m}(s). \tag{4}$$

□

From (2), (4) and Lemma 1 we have proved

Proposition 2. $D_{F,F,m}(s)$ is a Dirichlet series which has a meromorphic continuation to **C**, possibly has a unique pole at $s = k$, and satisfies a functional equation

$$D^*_{F,F}(2k-2-s) = D^*_{F,F,N}(2k-2-s) = \text{a finite sum of } \frac{\pm n^s}{P(s)} D^*_{F,F,m}(s)$$

where m, n are natural numbers with $m|N$ and $P(s)$ is a finite product of $1 - \tilde{m}^{2(k-s)}$ with $\tilde{m}|m$.

□

Now we can use Landau's Theorem for $D_{F,F,m}(s)$'s, because $D_{F,F,m}(s)/(1-p^{2(k-s)})$ has non-negative coefficients and has a unique pole at $s = k$, hence it converge for $s > k$. Therefore we have

$$\sum_{n \le x} ||\phi_n||^2 = \left(\mathrm{Res}_{s=k} \frac{D_{F,F}(s)}{s}\right) x^k + O_\varepsilon(x^{k-4/9+\varepsilon}) \quad (\forall \varepsilon > 0)$$

where ϕ_n is the n-th Fourier-Jacobi coefficient of $F(Z)$. Taking $x = m$ and $x = m-1$ and subtracting, we find

$$||\phi_m||^2 \ll_{\varepsilon,F} m^{k-4/9+\varepsilon},$$

hence

$$||\phi_m|| \ll_{\epsilon,F} m^{k/2-2/9+\varepsilon} \quad (\forall \varepsilon > 0). \tag{5}$$

By Proposition 2 and (5), we obtain

$$c(n,r) \ll_{\varepsilon,k} (m + |D|^{1/2+\varepsilon})^{1/2} |D| m^{5/18+\varepsilon}.$$

Both sides of (1) are invariant if T is replaced by ${}^tUTU$ $(U \in GL_2(\mathbf{Z}))$. Hence we may assume that

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \quad m = \min T,$$

so that $a(T) = c(n,r)$. By reduction theory we have $m = \min T \le \frac{2}{\sqrt{3}}|D|^{1/2}$ and complete the proof of Theorem.

□

Remark.

1. When $N = 1$, the Rankin convolution series $D_{F,F}(s)$ is a linear combination of spinor zeta functions of Hecke eigen forms, as shown in [K-S]. In order to deduce estimates for eigenvalues of Hecke operators, we need find a relation between $D_{F,F,m}(s)$'s and spinor zeta functions.

2. When we generalize Kohnen's method to higher genus, we should cut Z as follows;

$$Z = \left(\begin{array}{ccc|c} * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & * \\ \hline * & \dots & * & \tau' \end{array}\right).$$

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