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Regularizing effects for a class of Hamilton-Jacobi equations

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The first author talked on the subject of the title. The following is the résumé of the talk.

1 Introduction

We show that there are three regularizing effects: Lipschitz regularization, (local) semi-concavity regularization effect, and $C^{1,1}_{loc}$ regularizing effect for the following class of time-dependent Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t}+y\cdot\nabla_x u+H(\nabla_y u)=0,\quad t>0,\quad (x,y)\in R^N\times R^N, \tag{1}$$

$$u(0,x,y)=u_0(x,y),\quad (x,y)\in R^n\times R^n, \tag{2}$$

where u_0 is a bounded, continuous function in $R^N\times R^N$; $\cdot$ stands for the scalar product in $R^N\times R^N$; $H(\cdot)$ is a continuous function from R^N to R, satisfying the following assumptions.

$$H(\cdot)\quad is\quad convex, nonnegative, H(0)=0,$$

(H)

$$\lim_{|p|\to\infty}\frac{H(p)}{|p|}=\infty,\quad as\quad |p|\to\infty.$$

Our study is based on the viscosity solution theory introduced by M.G.Crandall and P.-L.Lions in [2]. We say that there is a Lipschitz regularizing effect of (1) when the solution $u(t,x,y)$ of (1), (2) is Lipschitz continuous in $(x,y)\in R^N\times R^N$ for all $t>0$, for an arbitrary initial conditions $u_0(x,y)$ assumed to be bounded and continuous.

We say that there is a local semi-concavity regularizing effect of (1), when for an arbitrary continuous initial condition $u_0(x,y)$, the solution $u(t,x,y)$ of (1)-(2) is locally

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semi-concave in $(x,y) \in R^N \times R^N$, for all $t > 0$. That is for any $R > 0$, $t > 0$, there exists a number $C_{R,t} > 0$ such that

$$u(t,x,y) - C_{R,t}(|x|^2 + |y|^2)$$

is semi-concave in the ball $B_R(0,0) \in R^N \times R^N$. The local semi-concavity regularizing effect leads the $C^{1,1}_{loc}$ regularizing effect of (1), which means here that the solution $u(t,x,y)$ of (1), (2) belongs to $C^{1,1}_{loc}(R^N \times R^N)$ for all $t > 0$, if $u_0(x,y)$ is bounded below, continuous and convex in $(x,y) \in R^N \times R^N$. These kinds of regularizations were first shown by J.M.Lasry and P.-L.Lions in [4]. and by P.-L. Lions in [6].

All the regularization effects above come from the existence of the function $L(t,x,y)$ defined on $t > 0$, $(x,y) \in R^N \times R^N$ with which the following inf-convolution formula

$$u(t,x,y) = \inf_{(x',y') \in R^N \times R^N} \{u_0(x',y') + L(t, x - x' - ty', y - y')\}, \quad t > 0, \quad (x,y) \in R^N \times R^N, \tag{3}$$

yields a viscosity solution of (1), (2), that is the value function of the associated control problem ((5)-(6) below). More precisely, denoting by H^* the Frenchel transform of $H(p)$

$$H^*(p) = \sup_{q \in R^N} \{\langle p, q\rangle - H(q)\}, \quad p \in R^N, \tag{4}$$

we set

$$u(t,x,y) = \inf_{\alpha(\cdot) \in \mathcal{A}} \{\int_0^t H^*(\alpha(s))ds + u_0(x_\alpha(t), y_\alpha(t))\}, \quad t > 0, \quad (x,y) \in R^N \times R^N, \tag{5}$$

where $\mathcal{A}$ is the set of all measurable functions $\alpha(\cdot)$ from $[0,t]$ to R^N such that $H(\alpha(\cdot))$ is integrable in $[0,t]$; $(x_\alpha(s), y_\alpha(s))$ $\quad 0 \le s \le t$ is the solution of the ordinary differential equation

$$\frac{d}{ds}(x_\alpha(s), y_\alpha(s)) = (-y_\alpha(s), \alpha(s)), \quad s \ge 0, \tag{6}$$

$$(x_\alpha(0), y_\alpha(0)) = (x,y), \quad (x,y) \in R^N \times R^N.$$

It is worth remarking that the existence of the "kernel" $L(t,x,y)$ in (3) comes from the controlability of the system (6), namely (6) has the following property: for any $t > 0, (x', y') \in R^N \times R^N$, there exists $\alpha(\cdot) \in \mathcal{A}$ for which the solution $(x_\alpha(s), y_\alpha(s))$ $\quad 0 \le s \le t$ of (6) satisfies $(x_\alpha(t), y_\alpha(t)) = (x', y')$.

Some properties of $L(t,x,y)$: non-negativity, convexity, lower and upper bounds, invariance property etc. will be given in Theorem 2.

Then, we shall investigate the reguralty of $L(t,x,y)$ itself and $u(t,x,y)$ given in (3). As for the Lipschitz regularity, $L(t,x,y)$ is locally Lipschits continuous in $(x,y) \in R^N \times R^N$ for any $t > 0$ (which we do not give the proof here) and $u(t,x,y)$ is Lipschitz continuous in $(x,y) \in R^N \times R^N$ for any $t > 0$. These results will be given in Theorems 3, 4. We mention that when the initial function $u_0(x,y)$ in (2) is bounded uniformly continuous, since $u(t,x,y)$ given in (3) is Lipschitz continuous (Theorem 4), this is the unique viscosity solution of (1)-(2) in the framework of the bounded uniformly continuous functions in $R^N \times R^N$.(See M.G.Crandall, L.C.Evans and P.L.Lions [3].) As for the local semi-concave regilarity and $C^{1,1}_{loc}$ regularlity, we shall first show in Theorem 5 that if $H(p) \in C^{2,1}_{loc}(R^N)$ and if $H(p)$ satisfies additional conditions (see Theorem 5), then $L(t,x,y) \in C^{1,1}_{loc}(R^N \times R^N)$ for any $t > 0$, which leads the local semi-concavity regularizing effect of (1). (Corollary 1) Next, in Theorem 6, we shall show that $u(t,x,y)$ given in (3) with a convex, continuous and bounded below initial function $u_0(x,y)$ belongs to $C^{1,1}_{loc}(R^N \times R^N)$ for any $t > 0$. (Theorem 6) As an example, for the case of $H(p) = |p|^2$, we can compute $L(t,x,y)$ explicitely and watch that there is the $C^{1,1}_{loc}$ regularising effect. This result is analogous to [4].

Moreover, we add the uniqueness problem for (1): first with continuous initial conditions u_0 (possibly unbounded from above) in the framewok of the positive, continuous solutions (see Theorem 7); next with a singular initial condition (see Theorem 8). In particular, the second result gives a characterization of the kernel $L(t,x,y)$ as the unique solution of

$$\frac{\partial L}{\partial t} = y \cdot \nabla_x L + H(\nabla_y L), \quad t > 0, \quad (x,y) \in R^N \times R^N,$$
$$\lim_{t\downarrow} L(t,x,y) = 0, \quad (x,y) = (0,0), \qquad = \infty, \quad (x,y) \neq (0,0),$$

with the condition of $\widetilde{L} \geq 0$ and $L \in C^{0,1}_{loc}((0,\infty] \times R^N \times R^N)$. These will be done in Therems 5,6.

In the subsequent arguments, we use the notations $R, \quad N, \quad R^+$ for the sets of real, natural and positive real numbers respectively. We denote the norm in R, R^N and $R^N \times R^N$ by $|r|, |x|$ and $|(x,y)|$, $(r \in R, \quad x,y \in R^N)$ respectively without confusion; the distance between two points (x,y) and $(\hat{x},\hat{y})$ by $|(x,y) - (\hat{x},\hat{y})|$. For a positive number $R > 0$, we write $B_R(0), \quad B_R(0,0)$ for the sets $\{x \in R^N \quad | \quad |x| < R\}$, $\{(x,y) \in R^N \times R^N \quad | \quad |(x,y)| < R\}$ respectively. We denote by $(x_\alpha(s), y_\alpha(s))$ the solution of the ordinary differential equation (6); by $\mathcal{A}(t;x,y;x',y')$ the set of all measurable functions $\alpha(\cdot)$ from $[0,t]$ to R^N such that $H^*(\alpha(\cdot))$ is integrable on $[0,t]$ and $(x_\alpha(t), y_\alpha(t)) = (x',y')$. As remarked above, from the controlability of the system (6)

$$\mathcal{A}(t;x,y;x',y') \neq \emptyset, \quad t > 0, \quad (x,y) \in R^N \times R^N. \tag{7}$$

In the throughout of this paper, the solution of the equation is in the sense of the viscosity solution, and we refer [2], [6] for its definition.

2 Existence of the kernel for the inf-convolution

We show the existence of the kernel $L(t, x, y)$ for the inf-convolution formula (3).

Theorem 1 *Let the Hamiltonian $H(p)$ in (1) satisfy the assumption (H). Let $u_0(x, y)$ in (2) be bounded and uniformly continuous. Then, there exists a unique viscosity solution of (1)-(2) and it is given by*

$$u(t,x,y) = \inf_{(x',y')\in R^N \times R^N} \{u_0(x',y') + L(t, x - x' - ty', y - y')\}, \tag{3}$$

$$t > 0, \quad (x, y) \in R^N \times R^N,$$

where L(t,x,y) is a real-valued function defined for $t > 0, \quad (x, y) \in R^N \times R^N$ by

$$L(t,x,y) = \inf_{\alpha(\cdot)\in\mathcal{A}(t;x,y;0,0)} \int_0^t H^*(\alpha(s))ds. \tag{8}$$

Proof

We shall rewrite (1) by using the Frenchel transformation (4)

$$\frac{\partial u}{\partial t} + y \cdot \nabla_x u + \sup_{\alpha \in R^N} \{-\langle \alpha, \nabla_y u\rangle - H^*(\alpha)\} = 0, \tag{9}$$

$$t > 0, \quad (x, y) \in R^N \times R^N.$$

It is known (see [6]) that the viscosity solution of (1)-(2) is given by

$$u(t,x,y) = \inf_{\alpha(\cdot)\in\mathcal{A}} \{u(t-s, x_\alpha(s), y_\alpha(s)) + \int_0^s H^*(\alpha(s'))ds'\}, \quad 0 \le s \le t, \quad (x,y) \in R^N \times R^N, \tag{10}$$

$$0 \le s \le t, \quad (x, y) \in R^N \times R^N,$$

where $\mathcal{A}$ is the set of all measurable functions $\alpha(\cdot)$ from $[0, t]$ to R^N such that $H^*(\alpha(s'))$ is integrable on $s' \in [0, t]$. We shall prove that (10) is equivalent to (3) with L defined in (8). So, let $s = t$ in (10) and we have

$$u(t,x,y) = \inf_{(x',y')\in R^N\times R^N} \inf_{\alpha(\cdot)\in\mathcal{A}(t;x,y;x',y')} \{u_0(x',y') + \int_0^t H^*(\alpha(s))ds\}. \tag{11}$$

By comparing (11), to (3) with (8), we see that it is enough to prove

$$\mathcal{A}(t; x, y; x', y') = \mathcal{A}(t; x - x' - ty', y - y'; 0, 0). \tag{12}$$

Let $\alpha_1(s) \in \mathcal{A}(t; x, y; x', y')$, denote by $(x_{\alpha_1}(s), y_{\alpha_1}(s))$ the solution of the O.D.E. (6) with $\alpha = \alpha_1$, and solve the O.D.E.

$$\frac{d}{ds}(\hat{x}_{\alpha_1}(s), \hat{y}_{\alpha_1}(s)) = (-\hat{y}_{\alpha_1}(s), -\alpha_1(s)), \quad 0 \le s \le t,$$

$$(\hat{x}_{\alpha_1}(0), \hat{y}_{\alpha_1}(0)) = (x - x' - ty', y - y').$$

Then, $(\hat{x}_{\alpha_1}(s), \hat{y}_{\alpha_1}(s)) = (x_{\alpha_1}(s) - x', y_{\alpha_1}(s) - y')$, $0 \le s \le t$ and we have $(\hat{x}_{\alpha_1}(t), \hat{y}_{\alpha_1}(t)) = (0,0)$, that is $\alpha_1(\cdot) \in \mathcal{A}(t; x - x' - ty', y - y'; 0, 0)$. Conversely, let $\alpha_2(\cdot) \in \mathcal{A}(t; x - x' - ty', y - y'; 0, 0)$, denote by $(x_{\alpha_2}(s), y_{\alpha_2}(s))$ the solution of the O.D.E. (6) with $\alpha = \alpha_2$, $(x_{\alpha_2}(0), y_{\alpha_2}(0)) = (x - x' - ty', y - y')$, and solve the O.D.E.

$$\frac{d}{ds}(\hat{x}_{\alpha_2}(s), \hat{y}_{\alpha_2}(s)) = (-\hat{y}_{\alpha_2}(s), -\alpha_2(s)), \quad 0 \le s \le t,$$

$$(\hat{x}_{\alpha_2}(0), \hat{y}_{\alpha_2}(0)) = (x, y).$$

Then, $(\hat{x}_{\alpha_2}(s), \hat{y}_{\alpha_2}(s)) = (x_{\alpha_2}(s) + x' + ty' - sy', y_{\alpha_2}(s) + y')$, $0 \le s \le t$ and we have $(\hat{x}_{\alpha_2}(t), \hat{y}_{\alpha_2}(t)) = (x', y')$, that is $\alpha_2(\cdot) \in \mathcal{A}(t; x, y; x', y')$. Therefore, (12) is proved and we have proved that (3) and (10) are equivalent.

It is known that for the bounded, uniformly continuous initial function $u_0(x, y)$, the solution $u(t, x, y)$ of (1),(2) is unique in the framework of bounded, uniformly continuous functions. We shall show below in Theorem 4 that $u(t, x, y)$ defined in (5) is Lipshitz continuous in $(x, y) \in R^N \times R^N$. By admitting temporarily this fact, since $u(t, x, y)$ is clearly bounded, we have proved our assertion.

3 Properties of the kernel

In this section, we shall show some properties of the kernel $L(t, x, y)$ of the inf-convolution formula (3), which will be used later in section 4 to study the regularizing effects.

Theorem 2 *The function $L(t, x, y)$ given in (8) has the following properties.*
(i) (Non-negativity)

$$L(t, x, y) > 0, \quad (x, y) \in R^N \times R^N \backslash (0, 0);$$
$$L(t, 0, 0) = 0, \quad (x, y) = (0, 0).$$

(ii) (Inf-convolution) For any $t, s > 0$,

$$L(t+s, x, y) = \inf_{(x', y') \in R^N \times R^N} \{L(t, x', y') + L(s, x - x' - sy', y - y')\}, \quad (x, y) \in R^N \times R^N. \tag{13}$$

(iii) (Convexity) For any $t > 0$, $L(t, x, y)$ is convex in $(x, y) \in R^N \times R^N$: for any $0 \le k \le 1$,

$$L(t, kx + (1-k)x', ky + (1-k)y') \le kL(t, x, y) + (1-k)L(t, x', y'), \tag{14}$$

$$t > 0, \quad (x, y), (x', y') \in R^N \times R^N.$$

(iv) (Scaling invariance) For any $\lambda > 0$,

$$L(t,x,y) = \lambda^{-1} L(\lambda t, \lambda^2 x, \lambda y), \quad t > 0, \quad (x,y) \in R^N \times R^N. \tag{15}$$

(v) (Lower estimate)

$$L(t,x,y) \geq \max\{tH^*(\frac{x}{t^2}), \quad tH^*(\frac{y}{t})\} \quad t > 0, \quad (x,y) \in R^N \times R^N. \tag{16}$$

(vi) (Upper estimate)

$$L(t,x,y) \leq \frac{t}{2}\{H^*(\frac{4x-ty}{t^2}) + H^*(\frac{-4x+3ty}{t^2})\} \quad t > 0, \quad (x,y) \in R^N \times R^N. \tag{17}$$

Proof

(i) is obvious in view of the definition of $L(t,x,y)$ in (8).
(ii) The relationship (13) comes also from the definition of L (8), by noticing

$$\mathcal{A}(s;x,y;x',y') = \mathcal{A}(s;x,y;x-x'-sy',y-y') \quad s > 0, \quad (x,y),(x',y') \in R^N \times R^N,$$

which was shown in the proof of Theorem 1.
(iii) Let $(x,y) = (x_1,...,x_N,y_1,...,y_N), (x',y') = (x'_1,...,x'_N,y'_1,...,y'_N) \in R^N \times R^N$; let $\alpha = (\alpha_1,...,\alpha_N) \in \mathcal{A}(t;x,y;0,0), \alpha' = (\alpha'_1,...,\alpha'_N) \in \mathcal{A}(t;x',y';0,0)$. We shall denote $(x'',y'') = (kx+(1-k)x', ky+(1-k)y'), \quad \alpha'' = k\alpha+(1-k)\alpha'$, and solve the following O.D.E.

$$\frac{d}{ds}(x''_{\alpha''}(s), y''_{\alpha''}(s)) = (-y''_{\alpha''}(s), -\alpha''(s)), \quad 0 \leq s \leq t,$$
$$(x''_{\alpha''}(0), y''_{\alpha''}(0)) = (x'',y'').$$

Then, we have $(x''_{\alpha''}(t), y''_{\alpha''}(t)) = (0,0)$ and $\alpha'' \in \mathcal{A}(t;x'',y'';0,0)$. Thus, by the convexity of H^*,

$$L(t,x'',y'') \leq \int_0^t H^*(\alpha''(s))ds \leq \int_0^t kH^*(\alpha(s))ds + \int_0^t (1-k)H^*(\alpha'(s))ds.$$

Since α, α' are arbitrary we have proved

$$L(t,x'',y'') \leq kL(t,x,y) + (1-k)L(t,x',y')$$

(iv) For arbitrary fixed $(x,y) \in R^N \times R^N, \quad t > 0$, let $\alpha(\cdot) \in \mathcal{A}(t;x,y;0,0)$ and let $\lambda > 0$. Define $\beta(s) = \alpha(\frac{s}{\lambda})$ for $0 \leq s \leq \lambda t$ and solve

$$\frac{d}{ds}(x_\beta(s), y_\beta(s)) = (-y_\beta(s), -\beta(s)), \quad 0 \leq s \leq \lambda t,$$
$$(x_\beta(0), y_\beta(0)) = (\lambda^2 x, \lambda y).$$

Then,

$$y_\beta(s) = \lambda y - \int_0^s \beta(s')ds' = \lambda y - \lambda \int_0^{\frac{s}{\lambda}} \alpha(s')ds',$$

$$x_\beta(s) = \lambda^2(x) - \lambda ys + \lambda \int_0^s \int_0^{\frac{s'}{\lambda}} \alpha(s'')ds''ds' = \lambda^2 x - \lambda ys + \lambda^2 \int_0^{\frac{s}{\lambda}} \int_0^{s'} \alpha(s'')ds''ds',$$

and we have $(x_\beta(\lambda t), y_\beta(\lambda t)) = (0,0)$, $\beta(\cdot) \in \mathcal{A}(\lambda t; \lambda^2 x, \lambda y; 0, 0)$.
Similarly, we can prove that if $\{\beta(s)\}(0 \le s \le \lambda t) \in \mathcal{A}(\lambda t; \lambda^2 x, \lambda y; 0, 0)$, then $\alpha(s) = \beta(\lambda s)$, $0 \le s \le t$ belongs to $\mathcal{A}(t; x, y; 0, 0)$.
Now, let $\alpha(s)$, $0 \le s \le t$, $\beta(s)$, $0 \le s \le \lambda t$ related by $\beta(s) = \alpha(\frac{s}{\lambda})$. Since

$$\int_0^{\lambda t} H^*(\beta(s))ds = \int_0^{\lambda t} H^*(\alpha(\frac{s}{\lambda}))ds = \lambda \int_0^t H^*(\alpha(s))ds,$$

from the above argument we have proved

$$\lambda^{-1} L(\lambda t, \lambda^2 x, \lambda y) = L(t, x, y).$$

(v) For any $\alpha \in \mathcal{A}(t; x, y; 0, 0)$, it is easy to see that the following holds

$$x = \int_0^t s\alpha(s)ds, \qquad y = \int_0^t \alpha(s)ds.$$

For $0 \le t \le 1$, since H^* is convex and $H^*(0) = 0$, from Jensen's inequality,

$$H^*(x) = H^*(\int_0^t s\alpha(s)ds) \le \int_0^t H^*(s\alpha(s))ds$$

$$\le \int_0^t sH^*(\alpha(s))ds \le \int_0^t H^*(\alpha(s))ds,$$

$$H^*(y) = H^*(\int_0^t \alpha(s)ds) \le \int_0^t H^*(\alpha(s))ds$$

which leads to

$$max\{H^*(x), H^*(y)\} \le L(t, x, y), \quad 0 \le t \le 1, \quad (x, y) \in R^N \times R^N. \tag{18}$$

For $t > 1$, from the invariance of $L(t, x, y)$ (iii) and (18),

$$L(t, x, y) = tL(1, \frac{x}{t^2}, \frac{y}{t}) \ge \max\{tH^*(\frac{x}{t^2}), tH^*(\frac{y}{t})\}.$$

(vi) For arbitrary $(x, y) \in R^N \times R^N$, the following control

$$\alpha = 4x - y, \quad 0 \le s \le \frac{1}{2}; \qquad = -4x + 3y, \quad \frac{1}{2} \le s \le 1,$$

belongs to $\mathcal{A}(1; x, y; 0, 0)$. Thus from the definition of $L(t, x, y)$

$$L(1, x, y) \leq \frac{1}{2}\{H^*(4x - y) + H^*(-4x + 3y)\}.$$

By using the invariance of $L(t, x, y)$,

$$L(t, x, y) = tL(1, \frac{x}{t^2}, \frac{y}{t}) \leq \frac{t}{2}\{H^*(\frac{4x}{t^2} - \frac{y}{t}) + H^*(\frac{-4x}{t^2} + \frac{3y}{t})\}.$$

4 Regularising effect

In this section, we shall study Lipschitz regularising effect, local semi-concavity regularizing effect and $C^{1,1}_{loc}$ regularising effect of the inf-convolution formula (3). We begin with the Lipschitz regularising effect.

Theorem 3 *The function $L(t, x, y)$ defined in (8) is locally Lipschitz continuous in $(x, y) \in R^N \times R^N$: for any $R > 0$, $t > 0$, there exists a constant $M_R > 0$ such that*

$$L(t, x, y) - L(t, \hat{x}, \hat{y}) \leq \sup_{|z| \leq M_R + 1} \frac{1}{t} H^*(16z)|(x, y) - (\hat{x}, \hat{y})|, \tag{19}$$

$$f(x, y), (\hat{x}, \hat{y}) \in B_R(0, 0).$$

The constant M_R is given by

$$M_R = \sup\{r \in R^+ | \sup_{|z| \leq r} H^*(4z) \leq \sup_{|z| \leq R} H^*(42z)\}.$$

Proof

We do not give the proof. It will be done similarly as the proof of the following Theorem.

Theorem 4 *Let $u_0(x, y)$ be bounded and continuous in $R^N \times R^N$, $|u_0(x, y)| \leq M$, $(x, y) \in R^N \times R^N$. Then,*

$$u(t, x, y) = \inf_{(x', y') \in R^N \times R^N} \{u_0(x', y') + L(t, x - x' - ty', y - y')\} \tag{3}$$

$$t > 0, \quad (x, y) \in R^N \times R^N,$$

is Lipschitz continuous in $R^N \times R^N$ for any $t > 0$:

$$u(t, x, y) - u(t, \hat{x}, \hat{y}) \leq 2t \sup_{|z| \leq G_{M,t} + 1} H^*(\frac{4(t+1)z}{t^2})|(x, y) - (x', y')|, \tag{20}$$

$$(x, y), \quad (\hat{x}, \hat{y}) \in R^N \times R^N,$$

where

$$G_{M,t} = \sup\{r \in R^+ \mid \sup_{|z| \leq r} H^*(\frac{z}{t^2}), \quad \sup_{|z| \leq r} H^*(\frac{z}{t}) \leq \frac{M}{t}\}. \tag{21}$$

Proof

First, let $t > 0$, and assume that $(x,y),(\hat{x},\hat{y}) \in R^N \times R^N$ satisfy $|(x,y)-(\hat{x},\hat{y})| \leq 1$. Since,

$$u(t,\hat{x},\hat{y}) = \inf_{(x',y')\in R^N\times R^N}\{u_0(x',y') + L(t,\hat{x}-x'-ty',\hat{y}-y')\} \leq M,$$

denoting

$$\mathcal{B}_0 = \{(x',y') \mid L(t,\hat{x}-x'-ty',\hat{y}-y') \leq M\},$$

we can write

$$u(t,\hat{x},\hat{y}) = \inf_{(x',y')\in\mathcal{B}_0}\{u_0(x',y') + L(t,\hat{x}-x'-ty',\hat{y}-y')\}. \tag{22}$$

From the lower bound (16) on $L(t,x,y)$ which we seeked in Theorem 2, the set $\mathcal{B}_0$ is contained in the set

$$\{(x',y') \mid tH^*(\frac{\hat{x}-x'-ty'}{t^2}), tH^*(\frac{\hat{y}-y'}{t}) \leq M\}.$$

We denote

$$G_{M,t} = \sup\{r \in R^+ \mid \sup_{|z|\leq r} H^*(\frac{z}{t^2}), \quad \sup_{|z|\leq r} H^*(\frac{z}{t}) \leq \frac{M}{t}\},$$

and remark that $(x',y') \in \mathcal{B}_0$ satisfies

$$|\hat{x}-x'-ty'|, \quad |\hat{y}-y'| \leq G_{M,t}. \tag{23}$$

Next, by using the inf-convolution formula (3), there exists $(x',y') \in R^N \times R^N$ which satisfies (23) and

$$u(t,x,y) - u(t,\hat{x},\hat{y}) \leq L(t,x-x'-ty',y-y') - L(t,\hat{x}-x'-ty',\hat{y}-y') \tag{24}$$

$$= L(t,\hat{x}-x'-ty'+x-\hat{x},\hat{y}-y'+y-\hat{y}) - L(t,\hat{x}-x'-ty',\hat{y}-y').$$

We shall write

$$(x-\hat{x},y-\hat{y}) = (k_1,...,k_N,l_1,...,l_N),$$

and let $k = \max_{1\leq i\leq N}\{|k_i| \vee |l_i|\}$, $\quad w = (\frac{k_1}{k},...,\frac{k_N}{k})$, $\quad z = (\frac{l_1}{k},...,\frac{l_N}{k})$. Then, $0 < k \leq 1$,

$$(x-\hat{x},y-\hat{y}) = k(\frac{k_1}{k},...,\frac{k_N}{k},\frac{l_1}{k},...,\frac{l_N}{k}) = k(w,z),$$

$$|\frac{k_i}{k}|, \quad |\frac{l_i}{l}| \leq 1, \quad 1 \leq i \leq N.$$

Therefore, by the convexity of $L(t,x,y)$,

$$L(t,\hat{x}-x'-ty'+x-\hat{x},\hat{y}-y'+y-\hat{y})$$

$$= L(t, \hat{x} - x' - ty' + kw, \hat{y} - y' + kz)$$
$$\leq (1-k)L(t, \hat{x} - x' - ty', \hat{y} - y') + kL(t, \hat{x} - x' - ty' + w, \hat{y} - y' + z).$$

Inserting this inequality into (24), and by using the upper estimate on $L(t,x,y)$ and (23),

$$u(t,x,y) - u(t,\hat{x},\hat{y}) \leq kL(t, \hat{x} - x' - ty' + w, \hat{y} - y' + z)$$
$$\leq t\{H^*(\frac{4(\hat{x} - x' - ty' + w) - t(\hat{y} - y' + z)}{t^2})$$
$$+H^*(\frac{-4(\hat{x} - x' - ty' + w) - 3t(\hat{y} - y' + z)}{t^2})\}\frac{|(x,y) - (\hat{x},\hat{y})|}{2},$$

and we have

$$u(t,x,y) - u(t,\hat{x},\hat{y}) \leq t \sup_{|z|\leq G_{M,t}+1} H^*(\frac{4(t+1)z}{t^2}),$$

for any (x,y), $(\hat{x},\hat{y}) \in R^N \times R^N$ such that $|(x,y) - (\hat{x},\hat{y})| \leq 1$. We therefore deduce (20) from the above inequality.

Next, we study $C^{1,1}_{loc}$ regularity of the kernel $L(t,x,y)$ of the inf-convolution formula (5).

Theorem 5 *Let the Hamiltonian $H(p)$ in (1) belong to $C^{2,1}_{loc}(R^N)$ and assume that ith Frenchel transformation $H^*(p)$ is strictly convex. Then, $L(t,x,y)$ defined in (8) belongs to $C^{1,1}_{loc}(R^N \times R^N)$ for $t > 0$.*

Proof

First, we shall see that if $\hat{\alpha}(\cdot) \in \mathcal{A}(t;x,y;0,0)$ satisfies

$$\int_0^t \nabla H^*(\hat{\alpha}(s))\beta(s)ds = 0 \tag{25}$$

holds for any measurable function $\beta(s)$ from $[0,t]$ to R^N such that

$$\int_0^t \beta(s)ds = 0, \quad \int_0^t s\beta(s)ds = 0, \tag{26}$$

then,

$$L(t,x,y) = \int_0^t H^*(\hat{\alpha}(s))ds. \tag{27}$$

In fact, if (25) holds, from the strict convexity of $H^*(p)$:

$$H^*(\hat{\alpha}(s) + \beta(s)) > H^*(\hat{\alpha}(s)) + \nabla H^*(\hat{\alpha}(s)) \cdot \beta(s), \quad a.e. s \in [0,t]$$

implies that $\hat{\alpha}(\cdot)$ is a global minimizer of the functional $\int_0^t H^*(\alpha(s))ds$ among $\alpha(\cdot) \in \mathcal{A}(t;x,y;0,0)$. Moreover, it is the unique minimizer, because if there is another local minimizer, say $\gamma \in \mathcal{A}(t;x,y;0,0)$, since $meas\{s \in [0,t] \quad | \quad \hat{\alpha}(s) \neq \gamma(s)\} > 0$, the strict convexity of $H^*(p)$ leads

$$meas\{s \in [0,t] \quad | \quad H^*(\frac{\hat{\alpha}(s)+\gamma(s)}{2}) < \frac{1}{2}H^*(\frac{\hat{\alpha}(s)}{2}) + \frac{1}{2}H^*(\frac{\gamma(s)}{2})\} > 0,$$

$$\int_0^t H^*(\frac{\hat{\alpha}(s)+\gamma(s)}{2})ds < \int_0^t H^*(\hat{\alpha}(s))ds,$$

which contradicts to (25), for $\beta(s) = \hat{\alpha}(s) - \frac{1}{2}(\hat{\alpha}(s)+\gamma(s))$ satisfies (26). Thus, $\hat{\alpha}(s) = \gamma(s), \quad a.e. s \in [0,t]$.

Next, from the implicit function theorem and also the strict convexity of H, for any $(x,y) \in R^N \times R^N$, there exists $(a,b) \in R^N \times R^N$ such that

$$(x,y) = (\int_0^t s\nabla H(a+bs)ds, \int_0^t \nabla H(a+bs)ds), \tag{28}$$

and the mapping $(x,y) \to (a,b)$ is $C^{1,1}_{loc}$. So, we define

$$\hat{\alpha}(s) = \nabla H(a+bs), \quad s \in [0,t],$$

and easily we have

$$\nabla H^*(\hat{\alpha}(s)) = \nabla H^*(\nabla H(a+bs)) = a+bs, \quad s \in [0,t].$$

Therefore, $\hat{\alpha}(\cdot)$ satisfies (25) and (27) holds. From the regularity assumption on $H(p)$, we conclude the proof.

Theorem 5 leads the local semi-concavity regularizing effect of (1).

Corollary 1 *Let the Hamiltonian $H(p)$ satisfy the assumptions in Theorem 5. Then, for any continuous function $u_0(x,y)$ in $R^N \times R^N$, for any $R > 0$,*

$$u(t,x,y) = \inf_{(x',y') \in R^N \times R^N} \{u_0(x',y') + L(t, x-x'-ty', y-y')\} \tag{3}$$

$$t > 0, \quad (x,y) \in R^N \times R^N,$$

is locally semi-concave.

Proof

It is a direct result from the $C^{1,1}_{loc}$ regularity of $L(t,x,y)$ proved in Theorem 5.

The local semi-concavity regularizing effect of (1) leads the $C^{1,1}_{loc}$ regularizing effect with the convex initial functions.

Theorem 6 *Let the Hamiltonian $H(p)$ satisfy the assumptions in Theorem 5. Then, for any bounded from below, continuous, and convex function $u_0(x,y)$ in $R^N \times R^N$,*

$$u(t,x,y) = \inf_{(x',y')\in R^N\times R^N}\{u_0(x',y') + L(t, x - x' - ty', y - y')\} \tag{3}$$

$$t > 0, \quad (x,y) \in R^N \times R^N,$$

belongs to $C^{1,1}_{loc}(R^N \times R^N)$.

Proof

First, we remark that $u(t,x,y)$ is convex in $(x,y) \in R^N \times R^N$ for any $t > 0$. To see this, let (x,y), $(\hat{x},\hat{y}) \in R^N \times R^N$, $0 \le k \le 1$ be arbitrary. Let $\alpha(s)$, $\beta(s)$ be measurable functions from $[0,t]$ to R^N. Let $(x_\alpha(s), y_\alpha(s))$, $0 \le s \le t, (\hat{x}_\beta(s), \hat{y}_\beta(s))$, $0 \le s \le t$ be the solutions of the O.D.E. (8), with the initial values $(x_\alpha(0), y_\alpha(0)) = (x,y)$, $(\hat{x}_\beta(0), \hat{y}_\beta(0)) = (\hat{x},\hat{y})$ respectively.
We denote $(x_\alpha(t), y_\alpha(t)) = (x', y')$, $(\hat{x}_\beta(t), \hat{y}_\beta(t)) = (\hat{x}', \hat{y}')$, put $\gamma(s) = k\alpha(s) + (1-k)\beta(s)$, $0 \le s \le t$, and solve

$$\frac{d}{ds}(x_\gamma(s), y_\gamma(s)) = (-y_\gamma(s), -\gamma(s)), \quad 0 \le s \le t,$$

$$(x_\gamma(0), y_\gamma(0)) = (kx + (1-k)\hat{x}, ky + (1-k)\hat{y}).$$

Then since,

$$(x_\gamma(t), y_\gamma(t)) = (kx' + (1-k)\hat{x}', ky' + (1-k)\hat{y}'),$$

by the convexity of u_0 and $H^*(p)$,

$$u(t, kx + (1-k)\hat{x}, ky + (1-k)\hat{y}) \le u_0(kx' + (1-k)\hat{x}', ky' + (1-k)\hat{y}') + \int_0^t H^*(\gamma(s))ds$$

$$\le ku_0(x'y') + (1-k)u_0(\hat{x}',\hat{y}') + k\int_0^t H^*(\alpha(s))ds + (1-k)\int_0^t H^*(\beta(s))ds$$

$$\le k\{u_0(x',y') + \int_0^t H^*(\alpha(s))ds\} + (1-k)\{u_0(\hat{x}',\hat{y}') + \int_0^t H^*(\beta(s))ds\},$$

and

$$u(t, kx + (1-k)\hat{x}, ky + (1-k)\hat{y}) \le ku(t,x,y) + (1-k)u(t,\hat{x},\hat{y}),$$

for $\alpha(\cdot), \beta(\cdot)$ are arbitrary.

Therefore, from Corollary 1, there is a number $C_R > 0$ such that

$$u(t,x,y) + C_R(|x|^2 + |y|^2) \quad \text{is convex in} \quad B_R(0,0),$$

$$u(t,x,y) - C_R(|x|^2 + |y|^2) \quad \text{is concave in} \quad B_R(0,0).$$

These relationships yield the C^1 differentiability of $u(t,x,y)$, from standard results of convex analysis theory. Next, by using the same techniques as in [4], [6], since $L \in C^{1,1}_{loc}(R^N \times R^N)$, we have $u(t,x,y) \in C^{1,1}_{loc}(R^N \times R^N)$ for $t > 0$.

Example 1 *Let N=1, $H(p) = |p|^2$. Then, by using the argument in the proof of Theorem 5, we can explicitely compute*

$$L(t,x,y) = \frac{3x^2 - 3xyt + y^2t^2}{t^3},$$

and we can see directly that $u(t,x,y)$ given by the inf-convolution formula is semi-concave for any continuous initial condition $u_0(x,y)$, and that if we assume moreover that $u_0(x,y)$ is bounded from below and convex, $u(t,x,y)$ belongs to $C^{1,1}_{loc}(R \times R)$ for any $t > 0$ as in [4].

5 Uniqueness result and characterization of the kernel

In this section, we give the uniqueness result for the solution of (1), with possibly unbounded, continuous initial condition $u_0(x,y)$ in the framework of the continuous, positive solution in $(0,\infty) \times R^N \times R^N$. That is, the inf-convolution formula (3) gives a unique solution of (1) in this framework. This result is stated in Theorem 7. Next, we deduce from this fact a characterization of the kernel $L(t,x,y)$ given in (10) in terms of the partial differential equation (1) with a singular-valued initial condition. This will be shown in Theorem 8.

Theorem 7 *Let $u(t,x,y)$ be a continuous solution of*

$$\frac{\partial u}{\partial t} + y \cdot \nabla_x u + H(\nabla_y u) = 0, \quad t > 0, \quad (x,y) \in R^N \times R^N, \tag{29}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in R^N \times R^N.$$

Then, the following holds.

(i) For any arbitrary number $R > 0$,

$$u(t,x,y) = \inf_{\alpha(\cdot)}\{ \int_0^{\tau_R \wedge t} H^*(\alpha(s))ds + u(t-\tau_R, x_\alpha(\tau_R), y_\alpha(\tau_R))1_{(\tau_R \le t)} \tag{30}$$

$$+ u_0(x_\alpha(t), y_\alpha(t))1_{(\tau_R > t)}\}, \quad t > 0, \quad (x,y) \in B_R(0,0),$$

where for each $\alpha(\cdot)$, $\tau_R = \inf\{t \ge 0 \mid (x_\alpha(t), y_\alpha(t)) \notin \overline{B_R(0,0)}\}$.

(ii) If

$$u(t,x,y) \geq 0, \quad t \geq 0, \quad (x,y) \in R^N \times R^N, \tag{31}$$

then

$$u(t,x,y) = \inf_{(x',y')\in R^N\times R^N} \{u_0(x',y') + L(t, x-x'-ty', y-y')\}$$

$$t > 0, \quad (x,y) \in R^N \times R^N.$$

Proof

We do not give the proof of this Theorem here. See our paper to appear.

Theorem 8 *Let $L(t,x,y) \in C^{0,1}_{loc}((0,\infty)\times R^N \times R^N)$ be a solution of*

$$\frac{\partial \hat{L}}{\partial t} + y\cdot\nabla_x \hat{L} + H(\nabla_y \hat{L}) = 0, \quad t > 0, \quad (x,y) \in R^N \times R^N, \tag{32}$$

$$\lim_{t\downarrow 0} \hat{L}(t,x,y) = 0, \quad (x,y) = (0,0); \qquad = \infty, \quad (x,y) \neq (0,0), \tag{33}$$

and assume that $\hat{L}(t,x,y) \geq 0$, $t > 0$, $(x,y) \in R^N \times R^N$. Then,

$$L(t,x,y) = \hat{L}(t,x,y), \quad t > 0, \quad (x,y) \in R^N \times R^N$$

Proof

By using Theorem 7,

$$\hat{L}(h+t,x,y) = \inf_{(x',y')\in R^N\times R^N} \{\hat{L}(h,x',y') + L(t, x-x'-ty', y-y')\}, \quad (x,y) \in R^N \times R^N, \tag{34}$$

for any $h, t > 0$. Therefore, clearly

$$\hat{L}(h+t,x,y) \leq L(t,x,y), \qquad (x,y) \in R^N \times R^N,$$

holds for any $h,t > 0$ which leads to $\hat{L} \leq L$. Next, from (34), for any small $\varepsilon > 0$, $h > 0$, there exists $(x_h^\varepsilon, y_h^\varepsilon)$ such that

$$\hat{L}(h+t,x,y) + \varepsilon \geq \hat{L}(h, x_h^\varepsilon, y_h^\varepsilon) + L(t, x - x_h^\varepsilon - ty_h^\varepsilon, y - y_h^\varepsilon)$$

$$\geq L(t, x - x_h^\varepsilon - ty_h^\varepsilon, y - y_h^\varepsilon),$$

here we used $\hat{L} \geq 0$. We see from (33), (34) that $(x_h^\varepsilon, y_h^\varepsilon) \to (0,0)$ as $h \downarrow 0$. Therefore, the right-hand side of the above inequality tends to $L(t,x,y)$ as $h \downarrow 0$, and we have

$$\hat{L}(t,x,y) \geq L(t,x,y), \quad (x,y) \in R^N \times R^N.$$

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