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Nonlinear m-sectorial operators and time-dependent Ginzburg-Landau equations

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§0. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. We consider the following problem:

$$\begin{aligned} &\frac{\partial\Phi}{\partial t}-(\lambda+i\alpha)\Delta\Phi+(\kappa+i\beta)|\Phi|^{p-1}\Phi-\gamma\Phi=0, \\ &\frac{\partial\Phi}{\partial\nu}(x,t)=0 \qquad (x\in\partial\Omega, \quad t\geq 0), \\ &\Phi(x,0)=\Phi_0(x). \end{aligned} \tag{1}$$

Here $\lambda>0$, $\kappa>0$, $p>1$ and α, β, $\gamma\in\mathbf{R}$ are constants; ν is unit outward normal of $\partial\Omega$, $i=\sqrt{-1}$ and Φ is $\mathbf{C}$-valued. (1) is called the time-dependent *Ginzburg-Landau* equation when $p=3$ (see Temam [**5**]). Introducing the new unknown $u=e^{-\gamma t}\Phi(=v+iw)$, the problem (1) is written as

$$\begin{aligned} &\frac{\partial u}{\partial t}-(\lambda+i\alpha)\Delta u+(\kappa+i\beta)e^{(p-1)\gamma t}|u|^{p-1}u=0, \\ &\frac{\partial u}{\partial\nu}(x,t)=0 \qquad (x\in\partial\Omega, \quad t\geq 0), \\ &u(x,0)=u_0(x)\ (=\Phi_0(x)). \end{aligned} \tag{2}$$

For the mathematical setting we introduce complex Hilbert space $X=L^2(\Omega;\mathbf{C})$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and define operators A, B and time-dependent operators $A(t)$, $B(t)$ as follows:

$$\begin{aligned} &D(A):=\{u\in H^2(\Omega;\mathbf{C});\ \frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega\}, \\ &Au:=-\Delta v-i\Delta w \quad \text{for} \quad u=v+iw\in D(A), \\ &D(B):=\{u\in X;\ |u|^{p-1}u\in X\}, \end{aligned}$$

$$Bu := |u|^{p-1}u \text{ for } u \in D(B),$$
$$B(t)u := e^{(p-1)\gamma t}Bu \text{ for } u \in D(B(t)) := D(B),\ t \in [0,T],$$
$$D := D(A) \cap D(B),$$
$$A(t)u := (\lambda + i\alpha)Au + (\kappa + i\beta)B(t)u \text{ for } u \in D(A(t)) := D,\ t \in [0,T],$$

where $H^2(\Omega; \mathbf{C})$ is the usual Sobolev Hilbert space and $T > 0$ is arbitrary. Then the problem (2) is regarded as one of initial value problems for standard abstract evolution equations of the form

$$\text{(3)} \qquad \frac{du}{dt} + A(t)u = 0, \quad t \in [0,T], \qquad u(0) = u_0.$$

To solve (3) we can apply the theory of nonlinear evolution equations developed by Kato [**2**]. In fact, under some conditions for λ, κ, p, α, β, γ we can show that $A(t)$ $(t \in [0,T])$ is m-accretive in X (see Lemma 10) and $A(\cdot)$ satisfies the smoothness condition:

$$\|A(t)u - A(s)u\| \le C(T)|t-s|(1 + \|u\| + \|A(s)u\|), \text{ for } t,\ s \in [0,T],\ u \in D,$$

(see Lemma 11).

§1. The main theorem and its corollary

we obtain the following theorem.

Theorem. *Let* $\lambda > 0$, $\kappa > 0$, $p > 1$, $\dfrac{|\beta|}{\kappa} \le \dfrac{2\sqrt{p}}{p-1}$, $\lambda\kappa + \alpha\beta > |\lambda\beta - \alpha\kappa|$. *Then for any* $\Phi_0 \in D$, *there exists a unique global strong solution* $\Phi = \Phi(x,t)$, $(x,t) \in \Omega \times [0,\infty)$ *to the problem* (1) *in* X.

Put $\alpha = \beta = 0$ in the problem (1). Then we have

Corollary. *If* $\lambda > 0$, $\kappa > 0$, $p > 1$, *then for any* $\Phi_0 \in D$ *the problem*

$$\text{(4)} \qquad \begin{aligned} &\frac{\partial \Phi}{\partial t} - \lambda \Delta \Phi + \kappa |\Phi|^{p-1}\Phi - \gamma \Phi = 0, && (x,t) \in \Omega \times [0,\infty), \\ &\frac{\partial \Phi}{\partial \nu}(x,t) = 0, && (x,t) \in \partial\Omega \times [0,\infty), \\ &\Phi(x,0) = \Phi_0(x), && x \in \Omega. \end{aligned}$$

has a unique global strong solution $\Phi = \Phi(x,t)$.

§2. Proof of theorem

In this section we shall prove our main Theorem. The proof needs some lemmas. Throught this section, we assume that $\lambda > 0$, $\kappa > 0$, $p > 1$. It is well-known that the operator A is a nonnegative selfadjoint operator in X. So we can easily obtain

Lemma 1. *$(\lambda+i\alpha)A$ is m-accretive in X.*

In the next Lemma 2 which implies that B is a nonlinear sectorial operator, the constant $\frac{p-1}{2\sqrt{p}}$ is recently determined by [**3**].

Lemma 2([**3**]). *For any $u_1,\ u_2 \in D(B)$ we have*

$$|\mathrm{Im}(Bu_1 - Bu_2,\ u_1 - u_2)| \leq \frac{p-1}{2\sqrt{p}}\mathrm{Re}(Bu_1 - Bu_2,\ u_1 - u_2).$$

Since the operator B is sectorial like this, the accretiveness of B is preserved under a little rotation. So we can obtain

Lemma 3. *Let $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$. Then $(\kappa+i\beta)B$ is accretive in X (We can replace B by $B(t)$).*

Proof. Let $u_1,\ u_2 \in D(B)$. Then it follows from Lemma 2 that

$$\begin{aligned}&\mathrm{Re}\big((\kappa+i\beta)(Bu_1 - Bu_2), u_1 - u_2\big)\\ \geq&\kappa\mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2) - |\beta||\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)|\\ \geq&\kappa\Big(\frac{2\sqrt{p}}{p-1} - \frac{|\beta|}{\kappa}\Big)|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \geq 0. \quad \square\end{aligned}$$

Let $f \in X$ then for almost every $x \in \Omega$ the equation

$$z + |z|^{p-1}z = f(x) \ \text{ in } \mathbf{C}$$

has a unique solution $z = u(x)$ such that $|u(x)| \leq |f(x)|$. Therefore $u \in D(B)$ and we obtain the following lemma.

Lemma 4. *B is m-accretive in X (We can replace B by $B(t)$).*

For every $\varepsilon > 0$ we set

$$J_\varepsilon = (I + \varepsilon B)^{-1}, \quad B_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon).$$

Lemma 5. *Let* $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$. *Then* $(\kappa + i\beta)B_\varepsilon$ *is accretive in* X.

Proof. Let $v_1,\ v_2 \in X$. Then it follows from Lemma 3 that

$$\begin{aligned}
&\mathrm{Re}\big((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), v_1 - v_2\big)\\
=&\mathrm{Re}\big((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), J_\varepsilon v_1 - J_\varepsilon v_2\big)\\
+&\mathrm{Re}\big((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), (v_1 - J_\varepsilon v_1) - (v_2 - J_\varepsilon v_2)\big)\\
=&\mathrm{Re}\big((\kappa + i\beta)(B(J_\varepsilon v_1) - B(J_\varepsilon v_2)), J_\varepsilon v_1 - J_\varepsilon v_2\big)\\
+&\mathrm{Re}\big((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), \varepsilon(B_\varepsilon v_1 - B_\varepsilon v_2)\big)\\
\geq&\varepsilon\kappa\|B_\varepsilon v_1 - B_\varepsilon v_2\|^2 \geq 0. \quad \square
\end{aligned}$$

Lemma 6. $C^1(\Omega; \mathbf{C}) \cap X$ *is invariant under* $(I + \varepsilon B)^{-1}$.

Proof. For any $f = g + ih \in C^1(\Omega; \mathbf{C}) \cap X$ we know that the equation

$$u_\varepsilon(x) + \varepsilon|u_\varepsilon(x)|^{p-1}u_\varepsilon(x) = f(x)$$

has a unique solution $u_\varepsilon(x) = v_\varepsilon(x) + iw_\varepsilon(x) \in D(B)$. It remains to show that $u_\varepsilon(x) \in C^1(\Omega; \mathbf{C})$. This equation is rewritten in the form:

$$\begin{cases} v_\varepsilon(x) + \varepsilon\big(v_\varepsilon(x)^2 + w_\varepsilon(x)^2\big)^{\frac{p-1}{2}} v_\varepsilon(x) = g(x),\\ w_\varepsilon(x) + \varepsilon\big(v_\varepsilon(x)^2 + w_\varepsilon(x)^2\big)^{\frac{p-1}{2}} w_\varepsilon(x) = h(x). \end{cases}$$

Put

$$\begin{aligned}
F(x, v_\varepsilon, w_\varepsilon) &:= v_\varepsilon + \varepsilon\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-1}{2}} v_\varepsilon - g(x),\\
G(x, v_\varepsilon, w_\varepsilon) &:= w_\varepsilon + \varepsilon\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-1}{2}} w_\varepsilon - h(x).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial(F,G)}{\partial(v_\varepsilon, w_\varepsilon)} :=& \begin{vmatrix} \frac{\partial F}{\partial v_\varepsilon} & \frac{\partial F}{\partial w_\varepsilon} \\ \frac{\partial G}{\partial v_\varepsilon} & \frac{\partial G}{\partial w_\varepsilon} \end{vmatrix}\\
=&\{1 + \varepsilon\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-1}{2}}\}^2 + \{1 + \varepsilon\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-1}{2}}\}\\
&\times \varepsilon(p-1)\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-3}{2}}\big(v_\varepsilon^2 + w_\varepsilon^2\big) \geq 1.
\end{aligned}$$

Therefore we can apply the implicit function theorem. $\square$

Lemma 7. *Re*$(Au,\ B_\varepsilon u) \geq 0$ *for* $u \in D(A)$.

Proof. Put

$$\widetilde{D}(A) := \{f = g + ih \in C^2(\Omega; \mathbf{C}) \cap H^2(\Omega; \mathbf{C});\ \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

It suffices to show our lemma for $f = g + ih \in \widetilde{D}(A)$. We set

$$v_\varepsilon + iw_\varepsilon = (I + \varepsilon B)^{-1}(g + ih).$$

Then we have

$$\begin{aligned}\frac{\partial v_\varepsilon}{\partial x_j} &= \frac{1}{q}\{(1 + aw_\varepsilon^2 + b)\frac{\partial g}{\partial x_j} - aw_\varepsilon v_\varepsilon \frac{\partial h}{\partial x_j}\},\\ \frac{\partial w_\varepsilon}{\partial x_j} &= \frac{1}{q}\{ - av_\varepsilon w_\varepsilon \frac{\partial g}{\partial x_j} + (1 + av_\varepsilon^2 + b)\frac{\partial h}{\partial x_j}\},\end{aligned}$$

where

$$a = \varepsilon(p-1)\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-3}{2}},\ b = \varepsilon\big(v_\varepsilon^2 + w_\varepsilon^2\big)^{\frac{p-1}{2}},\ q = (b+1)^2 + a(b+1)(v_\varepsilon^2 + w_\varepsilon^2).$$

It follows from this relation that

$$\begin{aligned}&\mathrm{Re}(Af,\ B_\varepsilon f)\\ =&\mathrm{Re}\big(- \Delta g - i\Delta h,\ \frac{1}{\varepsilon}[(g - v_\varepsilon) + i(h - w_\varepsilon)]\big)\\ =&\frac{1}{\varepsilon}\Big\{\int_\Omega \nabla g \cdot \nabla(g - v_\varepsilon)\ dx + \int_\Omega \nabla h \cdot \nabla(h - w_\varepsilon)\ dx\Big\}\\ =&\frac{1}{\varepsilon}\sum_{j=1}^N \int_\Omega \frac{\partial g}{\partial x_j} \cdot \frac{1}{q}\{q\frac{\partial g}{\partial x_j} - (1 + aw_\varepsilon^2 + b)\frac{\partial g}{\partial x_j} + aw_\varepsilon v_\varepsilon \frac{\partial h}{\partial x_j}\}\ dx\\ &+\frac{1}{\varepsilon}\sum_{j=1}^N \int_\Omega \frac{\partial h}{\partial x_j} \cdot \frac{1}{q}\{q\frac{\partial h}{\partial x_j} + av_\varepsilon w_\varepsilon \frac{\partial g}{\partial x_j} - (1 + av_\varepsilon^2 + b)\frac{\partial h}{\partial x_j}\}\ dx\\ \geq&\frac{1}{\varepsilon}\int_\Omega \frac{1}{q}\big[\{b^2 + b + ab(v_\varepsilon^2 + w_\varepsilon^2)\}(|\nabla g|^2 + |\nabla h|^2) + a(v_\varepsilon|\nabla g| - w_\varepsilon|\nabla h|)^2\big]\ dx\\ \geq&0. \quad \square\end{aligned}$$

Remark 8. Since X is a (complex) Hilbert space in our case, $B_\varepsilon u$ converges Bu $(u \in D(B))$ in X as $\varepsilon \downarrow 0$. Therefore we also have from Lemma 7 that

$$\mathrm{Re}(Au,\ Bu) \geq 0 \quad \text{for} \quad u \in D = D(A) \cap D(B).$$

Now we shall prove that the operator

$$A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$$

is m-accretive for every $t \in [0, T]$. Following the idea of T. Kato (see Brezis, Crandall and Pazy [1]), for every $f \in X$ we consider the approximate equations:

$$Au_\varepsilon + \frac{\kappa + i\beta}{\lambda + i\alpha} B_\varepsilon u_\varepsilon + u_\varepsilon = f, \quad \varepsilon > 0. \tag{5}$$

Since $(\lambda + i\alpha)A + (\kappa + i\beta)B_\varepsilon$ is m-accretive in X (for $\dfrac{|\beta|}{\kappa} \leq \dfrac{2\sqrt{p}}{p-1}$), (5) has a unique solution $u_\varepsilon \in D(A)$.

Lemma 9. *Let u_ε be the solution of* (5). *If $\lambda\kappa + \alpha\beta > 0$, then $\|B_\varepsilon u_\varepsilon\|$ is bounded for any $\varepsilon > 0$.*

Proof. It follows from (5) that

$$\mathrm{Re}(Au_\varepsilon, B_\varepsilon u_\varepsilon) + \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2}\|B_\varepsilon u_\varepsilon\|^2 + \mathrm{Re}(u_\varepsilon, B_\varepsilon u_\varepsilon) = \mathrm{Re}(f, B_\varepsilon u_\varepsilon)$$

Noting that $B0 = 0$ and

$$\begin{aligned}\mathrm{Re}(B_\varepsilon u_\varepsilon, u_\varepsilon) &= \mathrm{Re}\big(B(J_\varepsilon u_\varepsilon) - B0, J_\varepsilon u_\varepsilon - 0\big) + \mathrm{Re}(B_\varepsilon u_\varepsilon, \varepsilon B_\varepsilon u_\varepsilon) \\ &\geq \varepsilon \|B_\varepsilon u_\varepsilon\|^2 \geq 0,\end{aligned}$$

we have from lemma 7 that

$$\frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2}\|B_\varepsilon u_\varepsilon\|^2 \leq \|f\|\|B_\varepsilon u_\varepsilon\|. \quad \square$$

In (5), now it is routine work to prove that there exists a unique $u \in D$ such that

$u_\varepsilon \longrightarrow u$ strongly in X, $\quad Au_\varepsilon \longrightarrow Au$ weakly in X, $\quad Bu_\varepsilon \longrightarrow Bu$ weakly in X

as $\varepsilon \downarrow 0$. Hence we obtain

Lemma 10. *Let* $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$ *and* $\lambda\kappa + \alpha\beta > 0$. *Then* $(\lambda + i\alpha)A + (\kappa + i\beta)B$ *is m-accretive in* X. *The same is true for* $A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$ *for every* $t \in [0, T]$.

Lemma 11. *Let* $\lambda\kappa + \alpha\beta > |\lambda\beta - \alpha\kappa|$. *Then there exists a constant* $C = C(T) > 0$ *such that*

$$\|A(t)u - A(s)u\| \leq C|t-s|\|A(s)u\| \quad \textit{for} \quad t,\ s\ \in [0,T],\ \ u \in D.$$

Proof. Let $t,\ s \in [0,T]$ and $u \in D$. By the mean value theorem there exists a $C_1(T) > 0$ such that

$$\begin{aligned}\|A(t)u - A(s)u\| &= \|(\kappa + i\beta)(e^{(p-1)\gamma t} - e^{(p-1)\gamma s})Bu\| \\ &\leq C_1(T)|t-s|\|B(s)u\|.\end{aligned}$$

On the other hand we know from Remark 8 that

$$\operatorname{Re}(Au,\ Bu) \geq 0 \quad \text{for} \quad u \in D.$$

From this inequality we see that

$$\begin{aligned}\|\frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u\|^2 &\leq \operatorname{Re}(Au,\ \frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u) + \|\frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u\|^2 \\ &\leq \|Au + \frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u\| \cdot \|\frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u\|,\end{aligned}$$

and hence

$$\begin{aligned}\frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}\|B(s)u\| &\leq \|Au + \frac{\lambda\kappa+\alpha\beta}{\lambda^2+\alpha^2}B(s)u\| \\ &\leq \|Au + \frac{\kappa+i\beta}{\lambda+i\alpha}B(s)u\| + \frac{|\lambda\beta-\alpha\kappa|}{\lambda^2+\alpha^2}\|B(s)u\|.\end{aligned}$$

So we have

$$\|B(s)u\| \leq \frac{|\lambda - i\alpha|}{(\lambda\kappa+\alpha\beta) - |\lambda\beta-\alpha\kappa|}\|(\lambda+i\alpha)Au + (\kappa+i\beta)B(s)u\|$$

Thus we obtain

$$\|A(t)u - A(s)u\| \leq \frac{C_1(T)|\lambda - i\alpha|}{(\lambda\kappa+\alpha\beta) - |\lambda\beta-\alpha\kappa|}|t-s|\|A(s)u\|. \quad \square$$

Now we are in a position to prove our Theorem.

Proof of Theorem (completed). Since the domain D of $A(t)$ is independent of $t\in[0,T]$ and $A(t)$ is m-accretive in X, by Lemma 11 we can apply Kato's Theorem ([**2**]). Noting that $T>0$ is arbitrary, the solution $\Phi(x,t)$ to (1) exists for $(x,t)\in\Omega\times[0,\infty)$.

Added after the conference. In our Theorem, we can weaken the assumption. Namely our Theorem is still true when the condition $\lambda\kappa+\alpha\beta>|\lambda\beta-\alpha\kappa|$ is replaced by $\lambda\kappa+\alpha\beta>0$ (see [**6**]).

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