THE RATIO OF TWO NORMS OF QUADRATIC DIFFERENTIALS

松崎 克彦 (KATSUHIKO MATSUZAKI)

Department of Mathematics, Ochanomizu University

Let $T(g, n)$ be the Teichmüller space of the hyperbolic structures of finite area on a surface of genus $g$ and $n$ punctures $(2g - 2 + n > 0)$. We denote a Riemann surface corresponding to $\rho \in T(g, n)$ by $R_\rho$, a finitely generated Fuchsian group of the first kind uniformizing $R_\rho$ by $\Gamma(\rho)$.

Let $A(R_\rho)$ be the finite dimensional vector space of all the holomorphic quadratic differentials that may have a simple pole at each puncture. Every element $\varphi \in A(R_\rho)$ has finite $L^1$-norm $\|\varphi\|_1 = \iint_{R_\rho} |\varphi|$, and the Banach space with this norm is denoted by $A^1(R_\rho)$. On the other hand, $\varphi$ has finite $L^\infty$-norm $\|\varphi\|_\infty = \sup_{R_\rho} \rho^{-2} |\varphi|$, where $\rho$ also means the density of the hyperbolic metric, and this Banach space is denoted by $A^\infty(R_\rho)$.

The identity map of $A(R_\rho)$ defines a bounded linear operator $\iota_\rho$ from $A^1(R_\rho)$ to $A^\infty(R_\rho)$. We call the operator norm of $\iota_\rho$ the distortion index of the Riemann surface $R_\rho$ and denote it by $\kappa(\rho)$. Namely,

$$
\kappa(\rho) = \sup\{\|\varphi\|_\infty \mid \varphi \in A(R_\rho), \|\varphi\|_1 = 1\}.
$$

In [1], we have seen that $\kappa(\rho)$ is useful for the comparison of hyperbolic and extremal lengths on $R_\rho$ (Appendix). In this note, we supplement the following result.

**Theorem***. The map $\kappa : T(g, n) \to \mathbb{R}_+$ is continuous.

**Proof.** Assume that a sequence $\{\rho_m\}$ converges to $\rho$ in $T(g, n)$. We will show that $\lim_{m \to \infty} \kappa(\rho_m) \leq \kappa(\rho)$ first and $\lim_{m \to \infty} \kappa(\rho_m) \geq \kappa(\rho)$ second.

*This result will not appear elsewhere.*
First we choose $\varphi_m \in A^1(R_{\rho_m})$ for each $m$ such that $\|\varphi_m\|_1 = 1$ and $\|\varphi_m\|_\infty = \kappa(\rho_m)$. Lifting $\varphi_m$ to the unit disk $\Delta$, we may regard it as a holomorphic function on $\Delta$ which satisfies $\varphi_m(\gamma(z))\gamma'(z)^2 = \varphi_m(z)$ for any $\gamma \in \Gamma(\rho_m)$. From $\|\varphi_m\|_1 = 1$, we see that the family $\{\varphi_m\}$ is locally uniformly bounded, and hence it constitutes a normal family. Let $\varphi$ be any limit function of this family. It satisfies the automorphic condition for $\Gamma(\rho)$. Further, we easily see that $\|\varphi\|_1 = 1$, and hence $\|\varphi\|_\infty \leq \kappa(\rho)$. When $\varphi_{m_j}$ converge to $\varphi$ locally uniformly, $\kappa(\rho_{m_j}) = \|\varphi_{m_j}\|_\infty$ converge to $\|\varphi\|_\infty$. Therefore the limit supremum is less than or equal to $\kappa(\rho)$.

Next we choose $\varphi \in A^1(R_\rho)$ such that $\|\varphi\|_1 = 1$ and $\|\varphi\|_\infty = \kappa(\rho)$. As in the previous paragraph, we regard $\varphi$ as an automorphic function for $\Gamma(\rho)$. Then, by surjectivity of the Poincaré series operator, there is a holomorphic function $\psi$ such that $\|\psi\|_\Delta := \int\int_\Delta |\psi(z)| dx dy < \infty$ and

$$
\Theta_{\Gamma(\rho)} \psi := \sum_{\gamma \in \Gamma(\rho)} \psi(\gamma(z))\gamma'(z)^2 = \varphi .
$$

Using this $\psi$ and the Poincaré series operator $\Theta_{\Gamma(\rho_m)}$ for each $m$, we define an automorphic function for $\Gamma(\rho_m)$ by $\varphi_m = \Theta_{\Gamma(\rho_m)} \psi$. It is known that $\|\varphi_m\|_1 \leq \|\psi\|_\Delta$. Since $\sum_{\gamma \in \Gamma(\rho_m)} |\psi(\gamma(z))\gamma'(z)^2|$ converges locally uniformly with respect to $z$ and uniformly $m$, we can easily see that $\varphi_m = \Theta_{\Gamma(\rho_m)} \psi$ converge to $\varphi = \Theta_{\Gamma(\rho)} \psi$ locally uniformly. Therefore $\|\varphi_m\|_\infty / \|\varphi_m\|_1 (\leq \kappa(\rho_m))$ converge to $\|\varphi\|_\infty / \|\varphi\|_1 = \kappa(\rho)$. This implies that the limit infimum of $\kappa(\rho_m)$ is more than or equal to $\kappa(\rho)$. 

\[ \square \]

**Appendix: A summary of [1]**

Let $R$ be a topological surface not necessarily of finite type. Let $[\alpha]$ be a free homotopy class of a simple closed curve $\alpha$ in $R$ not contractible to a point nor a puncture, though puncture is definite after a metric is given. We denote the set of all such classes $\{[\alpha]\}$ by $S_R$. Providing a hyperbolic metric $\rho$ with $R$, we define the hyperbolic length $l_\rho(\alpha)$ of a homotopy class of $\alpha$ by the infimum of lengths of curves in $[\alpha]$ with respect to the hyperbolic metric $\rho$. On the other hand, the extremal length of the homotopy class of $\alpha$ is by definition

$$
E_\rho(\alpha) = \sup_{\sigma} \frac{(\inf_{\alpha \in [\alpha]} \int_\alpha \sigma(z)|dz|^2)^2}{\int_{R_\rho} \sigma(z)^2|dz|^2} ,
$$
where the supremum is taken over all Borel measurable conformal metrics $\sigma(z)|dz|$ on $R_\rho$.

We consider the ratio $E_\rho(\alpha)/l_\rho(\alpha)^2$. The value we are interested in is its upper bound, namely,

$$\nu(\rho) = \sup \left\{ \frac{E_\rho(\alpha)}{l_\rho(\alpha)^2} \mid [\alpha] \in S_R \right\}.$$

We estimate $\nu(\rho)$ using $\kappa(\rho)$ and a value $\lambda(\rho) = \inf_{[\alpha] \in S_R} l_\rho(\alpha)$ ($\lambda(\rho) = \infty$ for $S_R = \emptyset$).

We have the following result.

**Theorem A.** There exist universal constants $r_0$ and $r_1$ such that for an arbitrary hyperbolic Riemann surface $R_\rho$,

$$\frac{1}{\pi \lambda(\rho)} \leq \nu(\rho) \leq \kappa(\rho) \leq \max \{ \frac{r_0}{\lambda(\rho)}, r_1 \}.$$

If $R_\rho$ is of finite area, then there is a constant $r$ depending only on the Euler characteristic of $R$ such that

$$\frac{1}{\pi \lambda(\rho)} \leq \nu(\rho) \leq \kappa(\rho) \leq \frac{r}{\lambda(\rho)}.$$

**Proof.** A proof is done by combination of the following three claims.

**Claim 1** (Jenkins and Strebel). For an arbitrary Riemann surface $R_\rho$ and a homotopy class $[\alpha] \in S_R$, there is a holomorphic quadratic differential $\varphi(z)dz^2$ on $R_\rho$ such that

$$E_\rho(\alpha) = \left( \inf_{[\alpha]} \int_\alpha |\varphi|^{\frac{1}{2}}|dz| \right)^2 \int_{R_\rho} |\varphi||dz|^2.$$

**Claim 2** (Lehner+$\epsilon$). There exist universal constants $r_0$ and $r_1$ such that any holomorphic quadratic differential $\varphi$ for an arbitrary Fuchsian group $G$ satisfies

$$\|\varphi\|_\infty \leq \max \{ \frac{r_0}{\inf l_g}, r_1 \} \|\varphi\|_1,$$

where $l_g$ is the translation length of $g$ and the infimum is taken over all the hyperbolic elements of $G$. 
Claim 3 (Maskit+$\epsilon$). For any $[\alpha] \in S_R$ of an arbitrary hyperbolic Riemann surface $R_\rho$, we have

$$\frac{1}{\pi} \leq \frac{E_\rho(\alpha)}{l_\rho(\alpha)}.$$ 

The first inequality of Theorem A is known from Claim 3, and the third from Claim 2. Now we have only to show the second. Let $\varphi$ be the holomorphic quadratic differential with $\|\varphi\|_1 = 1$ which attains the extremal length $E_\rho(\alpha)$ as in Claim 1. Let $\alpha_0$ be the hyperbolic geodesic in $[\alpha]$. Then we see

$$E_\rho(\alpha)^{1/2} \leq \int_{\alpha_0} |\varphi(z)|^{1/2}|dz| = \int_{\alpha_0} (\rho^{-1}(z)|\varphi(z)|^{1/2})\rho(z)|dz|$$

$$\leq \|\varphi\|_\infty^{1/2} \int_{\alpha_0} \rho(z)|dz| \leq \kappa(\rho)^{1/2}l_\rho(\alpha).$$

This means that $\nu(\rho) \leq \kappa(\rho)$. □

There are several direct consequences from Theorem A.

**Corollary B** (Neibur-Sheingorn). For a hyperbolic Riemann surface $R_\rho$, the conditions $\kappa(\rho) < \infty$ and $\lambda(\rho) > 0$ are equivalent.

**Corollary C.** For a homotopy class $[\alpha] \in S_R$ of an arbitrary hyperbolic Riemann surface $R_\rho$,

$$E_\rho(\alpha) \leq \kappa(\rho)l_\rho(\alpha)^2.$$ 

**References**


Otsuka 2-1-1, Bunkyo-ku, Tokyo 112, Japan

E-mail address: matsuzak@math.ocha.ac.jp