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THE RATIO OF TWO NORMS OF QUADRATIC DIFFERENTIALS

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Let $T(g,n)$ be the Teichmüller space of the hyperbolic structures of finite area on a surface of genus g and n punctures $(2g-2+n>0)$. We denote a Riemann surface corresponding to $\rho\in T(g,n)$ by R_ρ, a finitely generated Fuchsian group of the first kind uniformizing R_ρ by $\Gamma(\rho)$.

Let $A(R_\rho)$ be the finite dimensional vector space of all the holomorphic quadratic differentials that may have a simple pole at each puncture. Every element $\varphi\in A(R_\rho)$ has finite L^1-norm $\|\varphi\|_1=\iint_{R_\rho}|\varphi|$, and the Banach space with this norm is denoted by $A^1(R_\rho)$. On the other hand, φ has finite L^∞-norm $\|\varphi\|_\infty=\sup_{R_\rho}\rho^{-2}|\varphi|$, where ρ also means the density of the hyperbolic metric, and this Banach space is denoted by $A^\infty(R_\rho)$.

The identity map of $A(R_\rho)$ defines a bounded linear operator ι_ρ from $A^1(R_\rho)$ to $A^\infty(R_\rho)$. We call the operator norm of ι_ρ the distortion index of the Riemann surface R_ρ and denote it by $\kappa(\rho)$. Namely,

$$\kappa(\rho)=\sup\{\ \|\varphi\|_\infty\mid\varphi\in A(R_\rho),\ \|\varphi\|_1=1\}\ .$$

In [1], we have seen that $\kappa(\rho)$ is useful for the comparison of hyperbolic and extremal lengths on R_ρ (Appendix). In this note, we supplement the following result.

Theorem*. *The map $\kappa: T(g,n)\to\mathbb{R}_+$ is continuous.*

Proof. Assume that a sequence $\{\rho_m\}$ converges to ρ in $T(g,n)$. We will show that $\overline{\lim}_{m\to\infty}\kappa(\rho_m)\le\kappa(\rho)$ first and $\underline{\lim}_{m\to\infty}\kappa(\rho_m)\ge\kappa(\rho)$ second.

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*This result will not appear elsewhere.

First we choose $\varphi_m \in A^1(R_{\rho_m})$ for each m such that $\|\varphi_m\|_1 = 1$ and $\|\varphi_m\|_\infty = \kappa(\rho_m)$. Lifting φ_m to the unit disk Δ, we may regard it as a holomorphic function on Δ which satisfies $\varphi_m(\gamma(z))\gamma'(z)^2 = \varphi_m(z)$ for any $\gamma \in \Gamma(\rho_m)$. From $\|\varphi_m\|_1 = 1$, we see that the family $\{\varphi_m\}$ is locally uniformly bounded, and hence it constitutes a normal family. Let φ be any limit function of this family. It satisfies the automorphic condition for $\Gamma(\rho)$. Further, we easily see that $\|\varphi\|_1 = 1$, and hence $\|\varphi\|_\infty \leq \kappa(\rho)$. When φ_{m_j} converge to φ locally uniformly, $\kappa(\rho_{m_j}) = \|\varphi_{m_j}\|_\infty$ converge to $\|\varphi\|_\infty$. Therefore the limit supremum is less than or equal to $\kappa(\rho)$.

Next we choose $\varphi \in A^1(R_\rho)$ such that $\|\varphi\|_1 = 1$ and $\|\varphi\|_\infty = \kappa(\rho)$. As in the previous paragraph, we regard φ as an automorphic function for $\Gamma(\rho)$. Then, by surjectivity of the Poincaré series operator, there is a holomorphic function ψ such that $\|\psi\|_\Delta := \iint_\Delta |\psi(z)|dxdy < \infty$ and

$$\Theta_{\Gamma(\rho)}\psi := \sum_{\gamma\in\Gamma(\rho)} \psi(\gamma(z))\gamma'(z)^2 = \varphi \, .$$

Using this ψ and the Poincaré series operator $\Theta_{\Gamma(\rho_m)}$ for each m, we define an automorphic function for $\Gamma(\rho_m)$ by $\varphi_m = \Theta_{\Gamma(\rho_m)}\psi$. It is known that $\|\varphi_m\|_1 \leq \|\psi\|_\Delta$. Since $\sum_{\gamma\in\Gamma(\rho_m)} |\psi(\gamma(z))\gamma'(z)^2|$ converges locally uniformly with respect to z and uniformly m, we can easily see that $\varphi_m = \Theta_{\Gamma(\rho_m)}\psi$ converge to $\varphi = \Theta_{\Gamma(\rho)}\psi$ locally uniformly. Therefore $\|\varphi_m\|_\infty/\|\varphi_m\|_1$ $(\leq \kappa(\rho_m))$ converge to $\|\varphi\|_\infty \,/\, \|\varphi\|_1 = \kappa(\rho)$. This implies that the limit infimum of $\kappa(\rho_m)$ is more than or equal to $\kappa(\rho)$. □

Appendix: A summary of [1]

Let R be a topological surface not necessarily of finite type. Let $[\alpha]$ be a free homotopy class of a simple closed curve α in R not contractible to a point nor a puncture, though puncture is definite after a metric is given. We denote the set of all such classes $\{[\alpha]\}$ by $\mathcal{S}_R$. Providing a hyperbolic metric ρ with R, we define the hyperbolic length $l_\rho(\alpha)$ of a homotopy class of α by the infimum of lengths of curves in $[\alpha]$ with respect to the hyperbolic metric ρ. On the other hand, the extremal length of the homotopy class of α is by definition

$$E_\rho(\alpha) = \sup_\sigma \frac{\left(\inf_{\alpha\in[\alpha]} \int_\alpha \sigma(z)|dz|\right)^2}{\iint_{R_\rho} \sigma(z)^2|dz|^2} \, ,$$

where the supremum is taken over all Borel measurable conformal metrics $\sigma(z)|dz|$ on R_ρ.

We consider the ratio $E_\rho(\alpha)/l_\rho(\alpha)^2$. The value we are interested in is its upper bound, namely,

$$\nu(\rho) = \sup\{ \frac{E_\rho(\alpha)}{l_\rho(\alpha)^2} \mid [\alpha] \in \mathcal{S}_R \} .$$

We estimate $\nu(\rho)$ using $\kappa(\rho)$ and a value $\lambda(\rho) = \inf_{[\alpha]\in\mathcal{S}_R} l_\rho(\alpha)$ ($\lambda(\rho) = \infty$ for $\mathcal{S}_R = \emptyset$).

We have the following result.

Theorem A. *There exist universal constants r_0 and r_1 such that for an arbitrary hyperbolic Riemann surface R_ρ,*

$$\frac{1}{\pi\lambda(\rho)} \leq \nu(\rho) \leq \kappa(\rho) \ \leq \max\{\frac{r_0}{\lambda(\rho)}, r_1\} .$$

If R_ρ is of finite area, then there is a constant r depending only on the Euler characteristic of R such that

$$\frac{1}{\pi\lambda(\rho)} \leq \nu(\rho) \leq \kappa(\rho) \ \leq \frac{r}{\lambda(\rho)} .$$

Proof. A proof is done by combination of the following three claims.

Claim 1 (Jenkins and Strebel). *For an arbitrary Riemann surface R_ρ and a homotopy class $[\alpha] \in \mathcal{S}_R$, there is a holomorphic quadratic differential $\varphi(z)dz^2$ on R_ρ such that*

$$E_\rho(\alpha) = \frac{\left(\inf_{\alpha\in[\alpha]} \int_\alpha |\varphi|^{\frac{1}{2}}|dz|\right)^2}{\iint_{R_\rho} |\varphi||dz|^2} .$$

Claim 2 (Lehner+ϵ). *There exist universal constants r_0 and r_1 such that any holomorphic quadratic differential φ for an arbitrary Fuchsian group G satisfies*

$$\|\varphi\|_\infty \leq \max\{\frac{r_0}{\inf l_g}, r_1\} \|\varphi\|_1 ,$$

where l_g is the translation length of g and the infimum is taken over all the hyperbolic elements of G.

Claim 3 (Maskit+ϵ). *For any $[\alpha] \in \mathcal{S}_R$ of an arbitrary hyperbolic Riemann surface R_ρ, we have*

$$\frac{1}{\pi} \leq \frac{E_\rho(\alpha)}{l_\rho(\alpha)} .$$

The first inequality of Theorem A is known from Claim 3, and the third from Claim 2. Now we have only to show the second. Let φ be the holomorphic quadratic differential with $\|\varphi\|_1 = 1$ which attains the extremal length $E_\rho(\alpha)$ as in Claim 1. Let α_0 be the hyperbolic geodesic in $[\alpha]$. Then we see

$$\begin{aligned} E_\rho(\alpha)^{1/2} &\leq \int_{\alpha_0} |\varphi(z)|^{1/2}|dz| = \int_{\alpha_0} (\rho^{-1}(z)|\varphi(z)|^{1/2})\rho(z)|dz| \\ &\leq \|\varphi\|_\infty^{1/2} \int_{\alpha_0} \rho(z)|dz| \leq \kappa(\rho)^{1/2} l_\rho(\alpha) . \end{aligned}$$

This means that $\nu(\rho) \leq \kappa(\rho)$. □

There are several direct consequences from Theorem A.

Corollary B (Neibur-Sheingorn). *For a hyperbolic Riemann surface R_ρ, the conditions $\kappa(\rho) < \infty$ and $\lambda(\rho) > 0$ are equivalent.*

Corollary C. *For a homotopy class $[\alpha] \in \mathcal{S}_R$ of an arbitrary hyperbolic Riemann surface R_ρ,*

$$E_\rho(\alpha) \leq \kappa(\rho) l_\rho(\alpha)^2 .$$

References

1. K. Matsuzaki, *Bounded and integrable quadratic differentials: hyperbolic and extremal lengths on Riemann surfaces*, Geometric Complex Analysis (J. Noguchi et al., eds.), World Scientific, 1996.

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