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One and two generator subgroups of Möbius groups in several dimensions

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1 Introduction

As corollaries of the inequality that is called the Jørgensen's inequality, he showed the following two theorems ([5]).

THEOREM A. *A non-elementary subgoup* Γ *of* $SL(2,C)$ *is discrete if and only if every two generator subgroup of* Γ *is discrete.*

THEOREM B. *A non-elementary subgroup* Γ *of* $SL(2,R)$ *is discrete if and only if every one generator subgroup of* Γ *is discrete.*

Theorem A is an immediate consequence of the Jørgensen' inequality. Theorem B is a generalization of classical results of Nielsen and Siegel as the following:

PROPOSITION A (Nielsen, [7]). *If* Γ *is a non-abelian and purely hyperbolic subgroup of* $SL(2,R)$, *then* Γ *is discrete.*

PROPOSITION B (Siegel, [8]). *If a non-elementary subgroup* Γ *of* $SL(2,R)$ *fails to be discrete, then* Γ *contains elliptic elements arbitrary close to the identity.*

Obviously we can see that Proposition B is a generalization of Proposition A. Applying Theorem A, Proposition B and the Selberg's theorem on finitely generated matrix groups will complete the proof of Theorem B. Note that Theorem B is not valid for $SL(2,C)$. In [4], Greenberg constructed a non-discrete, non-elementary subgroup of $SL(2,C)$, in which every element is loxodromic. In this note, we report some generalizations of both theorems to heigher dimensional case due to [1], [3], [4], and [6].

2 Preliminaries

For $n = 2, 3, \cdots$, we denote R^n and $\overline{R^n}$ by the n-dimensional Euclidean space and its one-point compactification, respectively. The unit ball B^n in R^n with the metric derived from the differential $ds^2 = 4dx^2/(1 - |x|^2)^2$ is a model of the n-dimensional hyperbolic space. The full Möbius group $M(\overline{R^n})$ is the group of Möbius transformations of $\overline{R^n}$, which is generated by inversions in spheres and reflections in planes. Let $M(B^n)$ be the subgroup of $M(\overline{R^n})$ which keeps B^n invariant. Then $M(B^n)$ is the group of hyperbolic isometries of B^n. Any element f of $M(B^n)$ is extended to a conformal automorphism of $cl(B^n)$, the closure of B^n. According to the Brower fixed point theorem, f has at least one fixed point in B^n or on its boundary $\partial B^n = S^{n-1}$. If there is a fixed point in B^n, we shall call f elliptic. If there is exactly one fixed point on ∂B^n, we shall call f parabolic. If there are exactly two fixed points on ∂B^n, f is called loxodromic. Let f be a loxodromic element. Then the hyperbolic geodesic connecting its fixed points is called the axis of f and denoted by σ_f. A loxodromic element f is called hyperbolic if every hyperbolic plane containing σ_f is f-invariant. Let f be an elliptic element and $x \in B^n$ its fixed point. Then there exists $g \in M(B^n)$ so that $g(x) = 0$ and gfg^{-1} fixes 0. It implies $gfg^{-1} \in O(n)$, the n-dimensional real orthogonal group. In general, an elliptic element may not have a fixed point on S^{n-1}. An elliptic element f has a fixed point on S^{n-1} if and only if it is conjugate to an orthogonal matrix which has 1 as an eigen-value. If n is odd and f is orientation-preserving, then f has a fixed point on S^{n-1}.

A subgroup Γ of $M(B^n)$ is called elementary if there exists a one or two point set on $cl(B^n)$ which is invariant under Γ. A non-elementary subgroup Γ is said to be n-dimensional on B^n if there is no proper hyperbolic subspace of B^n which is Γ-invariant. In other word, Γ is n-dimensional on B^n if and only if the convex core of Γ is n-dimensional as a manifold. Now we define a metric on $M(B^n)$. For $f, g \in M(B^n)$ we set

$$D(f, g) = \sup\{|f(x) - g(x)| \mid x \in S^{n-1}\},$$

where $| \ |$ denotes the Euclidean metric. Then $M(B^n)$ is a topological group with respect to the topology induced by the metric D. A subgroup Γ of $M(B^n)$ is discrete if it does not contain a sequence of elements converging to the identity with this topology.

Now we consider another model for hyperbolic geometry, the hyperboloid model. In this model, hyperbolic isometries are presented as the Lorentz matrices. Let

$$V = \{x = (x_0, x_1, \cdots, x_n) \in R^{n+1} \mid q(x, x) = 1\},$$

where $q(x, x) = x_0y_0 - x_1y_1 - \cdots - x_ny_n$ is the Lorentz form of the signature $(1, n)$. Then V is a two-sheeted hyperboloid. The subset $\{x \in V \mid x_0 > 0\}$ is one sheet of the hyperboloid V and denoted by V^+. Consider the Lorentz group

$$O(1, n; R) = \{A = (a_{ij})_{i,j=0,1,\cdots,n} \in M(n + 1, R) \mid q(Ax, Ax) = q(x, x)\},$$

in $n+1$-variables. Its subgroup $O^+(1,n;R) = \{A \in O(1,n;R) \mid a_{00} > 0\}$ of index 2 is called the future-preserving half of the Lorentz group. Then $O^+(1,n;R)$ is the group of the hyperbolic isometries of V^+ induced by the metric $ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$. Consider the mapping $F(x_0, x_1, \cdots, x_n) = (x_1/(1+x_0), \cdots, x_n/(1+x_0))$. Then F is an isometry between V^+ and B^n. Furthermore we have an isomorphism $\Phi : M(B^n) \longrightarrow O^+(1,n;R)$, where $\Phi(f) = F^{-1}fF, f \in M(B^n)$. It implies that the group $M(B^n)$ with the topology induced by D is isomorphic as a topological group to $O^+(1,n;R)$ with the natural topology. We identify $M(B^n)$ and $O^+(1,n;R)$ with this isomorphism Φ.

3 Theorems

First of all we consider a generalization of Theorem A to heigher dimensional cases by Martin and Abikoff-Haas.

THEOREM 1 ([1], [6]). *Let Γ be an n-dimensional subgroup of $M(B^n)$. Then Γ is discrete if and only if every two generator subgroup of Γ is discrete.*

Martin's proof of Theorem 1 is based on his generalization of the Jørgensen's inequality to $O^+(1,n;R)$ ([6], Theorem 4.5). Abikoff and Haas approach to Therem 1 in another way. The essential part of their argument is the following lemma which asserts the existence of a neighborhood of the identity in which discreteness or non-elementariness is violated.

LEMMA 1 ([1], [2]). *There exists a neighborhood U of the identity in $M(B^n)$ such that any discrete subgroup which is generated by elements of U is abelian.*

Now we study the next theorem and follow the proof.

THEOREM 2 ([1]). *Let n be an even number and Γ an n-dimensional subgroup of $M(B^n)$. Then Γ is discrete if and only if every one generator subgroup of Γ is discrete.*

Certain part of the proof of Theorem 2 depends on the following proposition due to Chen and Greenberg.

LEMMA 2. *Let Γ be an n-dimensional subgroup of $O^+(1,n;R)$. Suppose that there exists a non-empty open set $A \subset O^+(1,n;R)$ with $\Gamma \cap A \neq \emptyset$. Then Γ is discrete.*

The proof of this lemma is based on the theory of Lie groups. In order to prove Theorem 2, it suffices to show the existence of such an open set as in the lemma above for groups in which every elliptic element is finite order. We denote $E(n)$ by the set of all elliptic elements of $O^+(1,n;R)$ or $M(B^n)$. The following proposition is quite essential.

PROPOSITION 1. *Let n be an even number and U a neighborhood of the identity in $M(B^n)$. Then $U \cap E(n)$ contains an open set.*

PROOF. Let $\tilde{E}(n)$ be the subset of $E(n)$ consisting of elements which have not fixed points on $\partial B^n = S^{n-1}$. An elliptic element $f \in E(n)$ is contained in $\tilde{E}(n)$ if and only if f is conjugate to an orthogonal matrix which has not 1 as an eigen-value. Since n is even, we can easily see that $\tilde{E}(n) \cap U \neq \emptyset$ for any neighborhood U of the identity. Choose any $f_0 \in \tilde{E}(n)$ which is sufficiently close to the identity. Obviously we see $D(f_0, Id) = \sup\{|f_0(x) - x| \mid x \in S^{n-1}\} > 0$. Since f_0 has not a fixed point on S^{n-1} and S^{n-1} is compact, the quantity $\varepsilon_0 = \inf\{|f_0(x) - x| \mid x \in S^{n-1}\}$ is positive. To see this fact, suppose that ε_0 is zero. Then there exists $\{x_m\} \subset S^{n-1}$ such that $|f_0(x_m) - x_m| \searrow 0$ and $x_m \to x_0 \in S^{n-1}$ as $m \to \infty$. Hence we obtain $|f_0(x_0) - x_0| = 0$. It implies that f_0 has a fixed point $x_0 \in S^{n-1}$. It is a contradiction. So ε_0 is positive. Let g be any element of $M(B^n)$ which has a fixed point $\xi \in S^{n-1}$. That is to say g is one of a loxodromic, a parabolic, the identity or an elliptic element which has a fixed point on S^{n-1}. Then it follows $D(f_0, g) \doteq \sup\{|f_0(x) - g(x)| \mid x \in S^{n-1}\} \geq |f_0(\xi) - g(\xi)| = |f_0(\xi) - \xi| \geq \inf\{|f_0(x) - x| \mid x \in S^{n-1}\} = \varepsilon_0 > 0$. For any $\varepsilon \in (0,\ \varepsilon_0)$, we set $S(f_0) = \{f \in M(B^n) \mid D(f_0, f) < \varepsilon\}$. Note that $S(f_0)$ is open and $S(f_0) \cap \{M(B^n) - \tilde{E}(n)\} = \emptyset$. Thus $S(f_0)$ is an open set in $\tilde{E}(n)$ ($\subset E(n)$). Hence we obtain the required result.

q.e.d.

PROPOSITION 2 ([3], [4]). *Let n be an even number and Γ an n-dimensional subgroup of $O^+(1,n;R)$. Suppose that the identity is not approximated by elliptic elements of Γ. Then Γ is discrete.*

PROOF. Since the identity is not an accumulation point of elliptic elements of Γ, there exists a neighborhood U_0 of the identity in $O^+(1,n;R)$ which does not contain any elliptic element of Γ. By Proposition 1, there exists an open set A_0 in $U_0 \cap E(n)$ such that $A_0 \cap \Gamma = \emptyset$. Applying Lemma 2 will complete the proof of this proposition.

q.e.d.

Now we can prove Theorem 2. Assume that Γ is not discrete. Then, by Theorem 1, there exists a two generator subgroup Γ_0 of Γ which is non-discrete and non-elementary. By adjoining finitely many elements from Γ to Γ_0, we obtain a subgroup Γ_1 of Γ which is finitely generated, non-discrete and n-dimensional on B^n. Here we regard the group Γ_1 as a finitely generated group of matrices in $O^+(1,n;R)$. Then the Selberg's theorem yields that there exists a finite index normal subgroup $\tilde{\Gamma}_1$ of Γ_1 which is torsion-free. Since the index $[\Gamma_1 : \tilde{\Gamma}_1]$ is finite, $\tilde{\Gamma}_1$ is non-discrete and n-dimensional on B^n. By using Proposition 2, we conclude that $\tilde{\Gamma}_1$ contains an elliptic element of infinite order. The group generated by this elliptic element is not discrete. Hence we establish Theorem 2.

q.e.d.

Now we consider the odd-dimensional case. Let f be an elliptic element which is sufficiently colse to the identity. Since f has a fixed point on $\partial B^n = S^{n-1}$, f has a rotation axis σ in B^n. In other word, σ is a one-dimensional hyperbolic subspace which is pointwise fixed by f. For a sequence $\{\rho_m\}$ with $\rho_m \searrow 1$ $(m \to \infty)$, we denote d_m by a hyperbolic dilation (hyperbolic transformation) along σ with the translation length ρ_m ($m = 1, 2, \ldots$). Then the sequence $\{d_m f\}$ consists of distinct loxodromic transformations and $D(d_m f, f) \to 0$ as $m \to \infty$. It implies that in the odd-dimensional case, any elliptic element near the identity is an accumulation point of loxodromic elements. So we can not take an open set A in Lemma 2. In [4], Greenberg showed that Theorem 2 does extend to the odd-dimensional case by exhibiting a non-elementary, non-discrete subgroup of $SL(2,C)$ ($= M(B^3)$) in which every elemnt is loxodromic.

REMARK. The n-dimensionality condition on Γ in Theorem 1, 2 is quite essential. Abikoff and Haas constructed a non-elementary non-discrete subgroup Γ of $M(B^{2n})$ that leaves a 2-dimensional hyperbolic subspace of B^{2n} invariant and with the further property that every finitely generated subgroup of Γ is discrete.

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