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A Mean Value Theorem in Adele Geometry

Masanori MORISHITA (森下 昌紀)
Department of Mathematics, Faculty of Science
Kanazawa University (金沢大, 理)

"A Mean Vaule Theorem" in the title has its origin in the geometry of numbers created by H. Minkowski. In the following, I would like to explain how it has grown up according to the history of this beautiful topic.

The essential parts of our results were first obtained by Takao Watanabe independently. I wish to thank him for the correspondence on [M-W], in the course of which he kindly answered my question on algebraic groups and encouraged me to give a talk.

Finally, it is my great pleasure to dedicate this report to my teacher, Takashi Ono, who inspired me by a million of pleasant and fantastic conversations in the past years.

1. Minkowski-Hlawka's theorem

H. Minkowski asserted, in his letter to M. Hermite in 1893 ([Mi1]), the following statement. The proof was given by E. Hlawka about fifty years lalter, in 1944 ([H]).

Theorem 1 (Minkowski-Hlawka). *Let C be a star domain in the n-dimensional Euclidean space $\mathbf{R}^n, n \geq 2$, namely, C is a domain in $\mathbf{R}^n$ having the property that if $x \in C$, then $\lambda x \in C$ for $0 \leq \lambda \leq 1$. Assume that for any lattice L in $\mathbf{R}^n$ with $det(L) = 1$, C contains a point of L other than the origin.*

Then the Euclidean volume $vol(C)$ of C satisfies

$$vol(C) \geq \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots = \zeta(n),$$

where $\zeta(s)$ is the Riemann zeta function.

It is remarkable that Minkowski already predicted in [Mi2] that "Der Nachweis dieses Satzes erfordert eine arithmetische Theorie der Kontinuierlichen Gruppen aus allen linearen Transformationen." Actually, Hlawka gave a proof to Minkowski's assertion showing the following inequality:

For any $\epsilon > 0$, there exists $g \in SL_n(\mathbf{R})$ so that the following inequality holds for any bounded, compactly-supported, Riemannian-integrable function f on $\mathbf{R}^n$:

$$\int_{\mathbf{R}^n} f(x)dx + \epsilon \ \geq \sum_{z \in \mathbf{Z}^n \backslash \{0\}} f(gz),$$

where dx denotes the Euclidean volume element.

However, it remained unclear the relation to the fundamental domain of $SL_n(\mathbf{R})/SL_n(\mathbf{Z})$ which Minkowski had in his mind.

2. Siegel's mean value theorem

In 1945, C.L. Siegel ([Si]) refined Hlawka's inequality to an equality and realized Minkowski's original perspective.

Let dg be the invariant volume element on $SL_n(\mathbf{R})$ normalized by $vol_{dg}(SL_n(\mathbf{R})/SL_n(\mathbf{Z})) = 1$. Then, Siegel showed the following equality.

Theorem 2 (Siegel). *For any bounded, compactly-supported, Riemannian-integrable function f on $\mathbf{R}^n$, the following equality holds.*

$$\int_{\mathbf{R}^n} f(x)dx = \zeta(n) \int_{SL_n(\mathbf{R})/SL_n(\mathbf{Z})} (\textstyle\sum'_{z \in \mathbf{Z}^n \backslash \{0\}} f(gz))dg,$$

where the sum in the right hand side runs over $z = (z_1, \cdots, z_n) \in \mathbf{Z}^n \backslash \{0\}$ with G.C.D$(z_1, \cdots, z_n) = 1$.

Now, it is immediate to get Minkowski-Hlawka's theorem from Siegel's equality taking the characteristic function as f.

3. Weil's integration theory

Soon after Siegel ([Si]), A. Weil ([W1]) interpreted Siegel's mean value theorem as a Fubini-type theorem in his integration theory on a topological homogeneous space.

Let X be a topological space on which a locally compact unimodular group G acts transitively. Let L be a discrete subspace of X on which a discrete subgroup Γ of G acts stably. Let H be the stabilizer of $x \in L$ and $\gamma := H\cap$. Then, Weil showed the following Fubini-type equality.

Theorem 3 (Weil). *For all compactly-supported continuous function f on X,*

$$\int_X f(x)dx = vol(H/\gamma)^{-1} \int_{G/\Gamma} (\sum_{z \in \Gamma/\gamma} f(gz))dg,$$

where dx, dg are suitabl maching measures.

Clearly, we recover Siegel's theorem when it is applied to the situation: $G = SL_n(\mathbf{R}), \Gamma = SL_n(\mathbf{Z}), X = \mathbf{R}^n \backslash \{0\}, L = \mathbf{Z}^n \backslash \{0\}$.

As Siegel computed the volume of $SL_n(\mathbf{R})/SL_n(\mathbf{Z})$ using his mean value theorem, Weil used the above formula to compute the volume of G/Γ applying the Poisson summation formula. Later, he applied this idea to certain adele spaces to compute the Tamagawa numbers of classical algebraic groups ([W2]).

4. Ono's mean value theorem and Tamagawa number

T. Ono ([O2], 1968) took up the mean value theorem in the following adelic setting and applied his relative theory of Tamagawa numbers of semi-simple groups ([O1]).

Let X be a left homogeneous variety of a connected linear algebraic group G. Suppose that X, G and the action are defined over $\mathbf{Q}$ and $X(\mathbf{Q})$ is non-empty. Let H be the stabilizer of $x \in X(\mathbf{Q})$ and assume that H is connected. In the following, $\mathbf{A}$ denotes the adele ring of $\mathbf{Q}$. Then, Ono asked when the mean value theorem holds for the pairs

$$\begin{array}{ccc} G(\mathbf{A}) & \hookrightarrow & X(\mathbf{A}) \\ \cup & & \cup \\ G(\mathbf{Q}) & \hookrightarrow & X(\mathbf{Q}) \end{array}$$

and gave a sufficient condition for the mean value property of (G, X) in terms of the homotopy groups of the complex manifold $X(\mathbf{C})$.

To be precise, let $\omega_{\mathbf{A}}^G$ and $\omega_{\mathbf{A}}^H$ be the Tamagawa measures on $G(\mathbf{A})$ and $H(\mathbf{A})$, respectively ([O1]) and $\omega_{\mathbf{A}}^X$ be the canonical measure on $X(\mathbf{A})$ so that the maching $\omega_{\mathbf{A}}^G = \omega_{\mathbf{A}}^X \omega_{\mathbf{A}}^H$ holds. Assume further that G and H have no non-trivial $\mathbf{Q}$-rational characters and X is quasi-affine (X is quasi-projective if no condition is imposed on G and H and X is affine if H is reductive). Then, Ono introduced the following notions (Actually, Ono assumed that G and H has no non-trivial characters, but the above generalization is straightforward).

Definition 4.1. The homogeneous variety (G, X) is called *uniform* if there exists a constant $\tau(G, X)$ so that the following equality holds for any compactly supported continuous function f on $G(\mathbf{A})X(\mathbf{Q})$

$$\int_{G(\mathbf{A})X(\mathbf{Q})} f(x)\omega_{\mathbf{A}}^X = \tau(G, X)\tau(G)^{-1} \int_{G(\mathbf{A})/G(\mathbf{Q})} \Big(\sum_{z \in X(\mathbf{Q})} f(gz) \Big) \omega_{\mathbf{A}}^G,$$

where $\tau(G) = \int_{G(\mathbf{A})/G(\mathbf{Q})} \omega_{\mathbf{A}}^G$ is the Tamagawa number of G.

When that is so, the number $\tau(G, X)$ is called the *Tamagawa number* of a homogeneous variety (G, X) and we say that (G, X) has the *mean value property* if $\tau(G, X) = 1$.

Remark 4.2. If $G = X$ and the action is the left multiplication, the homogeneous space (G, G) is uniform and $\tau(G, G) = \tau(G)$ by thye Fubini theorem.

Siegel's case is, of course, $G = SL_n$ and $X = Aff^n \backslash \{0\}$.

To describe Ono's results (and ours in Section 6), we introduce the local and global classes in the rational points $X(\mathbf{Q})$. Let $y, z \in X(\mathbf{Q})$. We say that y is globally equivalent to z if there is $g \in G(\mathbf{Q})$ so that $y = gz$, and y is locally equivalent to z if there is $g_{\mathbf{A}} \in G(\mathbf{A})$ so that $y = g_{\mathbf{A}} z$. Thus, the local class containing z is $G(\mathbf{A})z \cap X(\mathbf{Q})$.

Then Ono showed, among other theorems, the following assuming that G, H have no rational characters, (G, X) is uniform, any local class consists of a global class, and that the Weil conjecture for the Tamagawa number of an algebraic group is true,

Theorem 4.3 ([O2]). *Under the above assumptions,*

$$\pi_1(X(\mathbf{C})) = \pi_2(X(\mathbf{C})) = 1 \Longrightarrow \tau(G, X) = 1.$$

Here, the Weil conjecture is the following:

Let G be a connected unimodular algebraic $\mathbf{Q}-$group. Then, $\pi_1(G(\mathbf{C})) = 1$ implies $\tau(G) = 1$.

Later, the Weil conjecture was settled by R.Kottwitz ([Ko3]) in 1988 assuming the Hasse principle for H^1 of the group of type E_8, and the latter was proved by Chernorsov in 1989 ([C]). The final formula is given as follows, where we use Borovoi's fundamental group which will be explained in the next section.

Theorem 4.4. *The Tamagawa number $\tau(G)$ of a unimodular connected linear $\mathbf{Q}$-group G is given by*

$$\tau(G) = \frac{[(\pi_1(G)_{\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})})_{tors}]}{[\mathrm{Ker}^1(\mathbf{Q}, G)]},$$

where $(\pi_1(G)_{\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})})_{tors}$ means the torsion part of the coinvariant quotient of $\pi_1(G)$ under $\mathrm{Gal}(\bar{k}/k)$, and $[*]$ means the cardinality of a set $*$, and

$$\mathrm{Ker}^1(\mathbf{Q}, G) := \mathrm{Ker}(H^1(\mathbf{Q}, G) \longrightarrow \prod_v H^1(\mathbf{Q}_v, G)).$$

5. Borovoi's fundamental group and abelian Galois cohomology

In this section, we introduce Borovoi's fundamental group and abelian Galois cohomology, which is a machinery to study the arithemetic of algebraic groups in a functorial way and so that of homogeneous varieties. For these matters, we refer to [B1], [B2], [B3], and also Appendix B to [Mil]. His results generalize Sansuc's results [Sa] on semisimple groups and refine Kottwitz's work ([Ko1],[Ko2]) on Galois cohomology using the Langlands group.

Let k be a field of characteristic zero and $\bar{k}$ a fixed algebraic closure of k. First, we assume that G is reductive. Let G^{ss} be the derived group of G and G^{sc} be the universal k-covering of G^{ss} ([O1], Appendix I). Consider the composition

$$\rho : G^{sc} \to G^{ss} \subset G.$$

Take a maximal torus T in $G_{\bar{k}}$ and put $T^{sc} = \rho^{-1}(T)$. We then define

$$\pi_1(G,T) := X_*(T)/\rho_* X_*(T^{sc}),$$

where $X_*(S)$ denotes the group of one-parameter subgroups of a torus S.
If T' is another maximal torus in $G_{\bar{k}}$, there is $g \in G(\bar{k})$ so that $T' = gTg^{-1} = Int(g)(T)$. Then, $Int(g)$ induces the isomorphism $g_* : \pi_1(G,T) \simeq \pi_1(G,T')$ which does not depend on the choice of g. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\pi_1(G,T)$ in the following way. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, there is $g_\sigma \in G(\bar{k})$ so that $T^\sigma = g_\sigma^{-1} T g_\sigma$. Then, σ acts on $\pi_1(G,T)$ as the composition

$$\pi_1(G,T) \xrightarrow{\sigma_*} \pi_1(G,T^\sigma) \xrightarrow{(g_\sigma)_*} \pi_1(G,T).$$

We see that the above isomorphism g_* is $\mathrm{Gal}(\bar{k}/k)$-equivariant. So, we simply write $\pi_1(G)$ for this Galois module. For a connected linear k-group G, we set $\pi_1(G) := \pi_1(G/G^u)$, where G^u is the unipotent radical of G, and call it *Borovoi's fundamental group* of G. Then, $\pi_1(\cdot)$ is an exact functor from the category of connected linear k-groups to $\mathrm{Gal}(\bar{k}/k)$-modules, finitely generated over $\mathbf{Z}$. One sees that an inner twisting $G \to G'$ induces the isomorphism $\pi_1(G) \simeq \pi_1(G')$, and that if $k \subset \mathbf{C}$, $\pi_1(G)$ is canonically isomorphic to the topological fundamental group of the complex Lie group $G(\mathbf{C})$ as abelian groups.

Next, we define the abelian Galois cohomology groups of a connected reductive group G by

$$H^i_{ab}(k,G) := \mathbf{H}^i(k, T^{sc} \to T) \quad (i \geq -1),$$

where $\mathbf{H}^i$ means the Galois hypercohomology of the complex

$$0 \to T^{sc} \to T \to 0,$$

where T^{sc} and T sit in degree -1 and 0, respectively.
Noting that $(X_*(T^{sc}) \xrightarrow{\rho_*} X_*(T)) \to \pi_1(G)$ is a short torsion free resolution of

$\pi_1(G)$ and that $S(\bar{k}) = X_*(S) \otimes \bar{k}^\times$ for a k-torus S, we can see that $H^i_{ab}(k,G)$ depends only on the Galois module $\pi_1(G)$. For a connected k-group G, we set $H^i_{ab}(k,G) := H^i_{ab}(k, G/G^u)$. On the other hand, for a connected reductive group G, we observe that $\rho : G^{sc} \to G$ is a crossed module of algebraic groups over k and so we can also define, in terms of cocycles, the hypercohomology

$$\mathbf{H}^i(k, G^{sc} \to G)$$

for $i = -1, 0, 1$, in a functorial way. Then, using the morphism $(1 \to G) \to (G^{sc} \to G)$ and the quasi-isomorphism $(T^{sc} \to T) \to (G^{sc} \to G)$ of crossed modules, we define the abelianization maps

$$ab^i : H^i(k, G) \longrightarrow H^i_{ab}(k, G)$$

for $i = 0, 1$ (For ab^2, see [B2]). For a connected k-group G, the abelianization maps are defined by the composition

$$H^i(k,G) \to H^i(k, G/G^u) \overset{ab^i}{\to} H^i_{ab}(k, G/G^u) = H^i_{ab}(k, G).$$

We note that if G is semisimple, $\pi_1(G) = \mathrm{Ker}(\rho)(-1)$ (Tate twist), $H^i_{ab}(k,G) = H^{i+1}(k, \mathrm{Ker}\rho)$ and ab^i $(i = 0, 1)$ are connecting homomorphisms attached to the exact sequence, $1 \to \mathrm{Ker}\rho \to G^{sc} \overset{\rho}{\to} G \to 1$.

Finally, we state Borovoi's theorem on the Tate-Shafarevich set $\mathrm{Ker}^1(\mathbf{Q}, G)$, where G is a connected linear $\mathbf{Q}-$group.

Theorem 5 ([B3]). *The abelianization map $ab^1 : H^1(\mathbf{Q}, G) \to H^1_{ab}(\mathbf{Q}, G)$ induces a bijection of* $\mathrm{Ker}^1(\mathbf{Q}, G)$ *onto the abelian group* $\mathrm{Ker}^1_{ab}(\mathbf{Q}, G)$*, which is functorial in G, where,*

$$\mathrm{Ker}^1_{ab}(\mathbf{Q}, G) := \mathrm{Ker}(\ H^1_{ab}(\mathbf{Q}, G) \longrightarrow \prod_v H^1_{ab}(\mathbf{Q}_v, G)).$$

6. Results

In this section, we assume as in Section 4 that G, H are connected linear algebraic groups having no non-trivial $\mathbf{Q}-$rational characters and X is quasi-affine.

Using the theorem 4.4 and some results stated in the section 5, we can show the followings. These were first obtained by Takao Watanabe independently in the case that G, H have no non-trivial characters (i.e., (G, X) is *special* in the sense of Ono [O2]). The key observation is that the fundamental group $\pi_1(G)$ does not change under an inner twisting. For the precise proofs, we refer to [M-W] and [Mo].

Theorem 6.1. *Any homogeneous variety (G, X) is uniform.*

Theorem 6.2. *The set of global classes in a local class containing $x \in X(\mathbf{Q})$ is canonically bijective to the finite abelian group*

$$\mathrm{Ker}(\mathrm{Ker}^1_{ab}(\mathbf{Q}, H) \longrightarrow \mathrm{Ker}^1_{ab}(\mathbf{Q}, G)).$$

In particular, it is independent of x.

Theorem 6.3. *The Tamagawa number of (G, X) is given by*

$$\tau(G, X) = \frac{[\pi_1(G)_{\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})}]}{[(\pi_1(H)_{\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})}][\mathrm{Coker}(\mathrm{Ker}^1(\mathbf{Q}, H) \to \mathrm{Ker}^1(\mathbf{Q}, G))]}$$

The following corollary is a generalization and refinement of Ono's theorem. The proof is immediate from the homotopy exact sequence attached to the fibration

$$1 \to H(\mathbf{C}) \to G(\mathbf{C}) \to X(\mathbf{C}) \to 1.$$

Corollary 6.4. *If the first two homotopy groups of the complex manifold $X(\mathbf{C})$ vanish, then $\tau(G, X)$ has the mean value property.*

Example 6.5 ([B-R],6.6). Let $f = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbf{Z}[t]$ be an irreducible polynomial. The group $G = SL_n$ acts on $X = \{x \in M_n \mid det(tI_n - x) = f(t)\}$ transitively by $(g, x) \mapsto g^{-1}xg$. The stabilizer H of the $\mathbf{Q}$-rational point

$$x = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdot & -a_n \\ 1 & 0 & 0 & \cdots & \cdot & -a_{n-1} \\ 0 & 1 & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$$

is the $\mathbf{Q}$-anisotropic torus $\mathrm{Ker}(N : R_{K/\mathbf{Q}}(\mathbf{G_m}) \to \mathbf{G_m})$, where $K = \mathbf{Q}(\alpha), f(\alpha) = 0$, and N is the norm map attached to $K/\mathbf{Q}$.

Then, by Theorem 6.3 and the claim 6.6.1 of [B-R] which computes $\pi_1(H)_{\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})} = H^{-1}(L/Q, X_*(H))$, if L is the Galois closure of $K/\mathbf{Q}$, we have

$$\tau(G, X) = [\mathrm{Coker}(\mathrm{Gal}(L/K)^{ab} \to \mathrm{Gal}(L/\mathbf{Q})^{ab})]^{-1},$$

where ab means the abelianization.

For example, if $\mathrm{Gal}(L/\mathbf{Q})$ is the symmetric group S_n $(n \geq 3)$, (G, X) has the mean value property.

Example 6.6 (Adelic Minkowski-Hlawka). Let C be a compact subset of $G(\mathbf{A})X(\mathbf{Q})$. We can take a characteristic function of C as f in the integration formula in (4.1) to get

$$vol_{\omega^X_{\mathbf{A}}}(C) = \tau(G, X)\tau(G)^{-1} \int_{G(\mathbf{A})/G(\mathbf{Q})} [gX(\mathbf{Q}) \cap C]\omega^G_{\mathbf{A}}.$$

Hence, we have a version of Minkowski-Hlawka theorem with a certain congruence condition:

If $vol_{\omega^X_{\mathbf{A}}}(C) \geq \tau(G, X)$, then there is $g \in G(\mathbf{A})$ so that X has a $\mathbf{Q}$−rational point in gC, and

if for any $g \in G(\mathbf{A})$, gC contains a point of $X(\mathbf{Q})$, then $vol_{\omega^X_{\mathbf{A}}}(C) \geq \tau(G, X)$.

7. A dream: Tamagawa characteristic class ?!

From the history of the mean value theorem, it is clear that the theory of Tamagawa numbers was originally, at least in its spirit, related to integral geometry, ergodic theory,... The resemblance between Ono-Kottwitz formula and Gauss-Bonnet formula is obvious. The formula given in Example 6.6 looks similar to some formulas in integral geometry such as Poincaré's one:

$$vol(C_2) = (4vol(C_1))^{-1} \int_{M(\mathbf{R}^2)} [gC_1 \cap C_2]dg,$$

where C_i are closed curves in $\mathbf{R}^2$ and $M(\mathbf{R}^2)$ is the group of motions on $\mathbf{R}^2(cf.[T])$.

After Gauss-Bonnet, Crofton, and Poincaré, one direction in which integral geometry has been deepen, as far as I understand, is such theories as the volume of a tube, characteristic classes due to Weyl, Chern, and Griffiths etc.(e.g.,see [G]).

It would be wonderful if there exist theories involving intermediate-dimensional subvarieties, characteristic classes,.., in the context of adele geometry ! (M. Kuga ([Ku]) posed a similar problem).

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