数理解析研究所講究録
972巻 1996年 68-77

高次局所誘導近似について

On a Higher-order Localized Induction Approximation

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The motion of a vortex filament in an inviscid incompressible fluid is described by the Biot-Savart law. The localized induction equation (LIE) is the simplest model to capture the leading-order behavior of this motion [1, 2]. Hasimoto [3] showed that the LIE is equivalent to the cubic nonlinear Schrödinger equation (NLS) for a complex variable, implying that the LIE is completely integrable. Magri [4] unveiled the bi-Hamiltonian structure that underlies this integrability and thereby manipulated a recursion operator to generate an infinite sequence of integrals in involution and commuting Hamiltonian vector fields. With the help of the connection between the NLS and the LIE, Langer and Perline [5] constructed a recursion operator to generate an infinite sequence of commuting vector fields associated with the LIE for a class of asymptotically linear curves. We call this sequence the Langer-Perline hierarchy (LPH). Let $\boldsymbol{X} = \boldsymbol{X}(s,t)$ be a point on the filament and $\boldsymbol{V}^{(n)} = \boldsymbol{V}^{(n)}(s,t)$ the n-th term of the LPH with s and t being the arclength and the time, respectively. They are then listed as follows:

$$\begin{aligned}
\boldsymbol{V}^{(1)} &= \boldsymbol{X}_s \times \boldsymbol{X}_{ss}\,, && (1)\\
\boldsymbol{V}^{(2)} &= \boldsymbol{X}_{sss} + \frac{3}{2}\boldsymbol{X}_{ss} \times (\boldsymbol{X}_s \times \boldsymbol{X}_{ss})\,, && (2)\\
&\vdots \\
\boldsymbol{V}^{(n)} &= -\boldsymbol{X}_s \times \boldsymbol{V}_s^{(n-1)} + \mathcal{T}^{(n)}\boldsymbol{X}_s\,, && (3)\\
&\vdots
\end{aligned}$$

where the subscripts denote the partial differentiation with respect to the indicated variables and $\mathcal{T}^{(n)}$ is a function to be determined by the condition of the arclength parametrization: $\boldsymbol{V}_s^{(n)} \cdot \boldsymbol{X}_s = 0$. Equaling $\boldsymbol{V}^{(1)}$ to $\boldsymbol{X}_t$, the first equation gives the LIE with appropriately rescaled time. Further, if we take $\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon\boldsymbol{V}^{(2)}$, ϵ some parameter, we recover the localized induction equation of a vortex filament with an axial flow in the core [6, 7]. Note that it is equivalent to the Hirota equation [8] which results from a summation of the first two terms of the NLS hierarchy as it should be. With this observation, it is tempting to pursue the summation procedure of the commuting vector fields of the LPH. Families of evolution equations that preserve local geometric invariants are produced by combining finite terms of the LPH [9, 10].

In this report, we consider the evolution equation of a curve obtained by summing up all of the terms of the LPH, namely,

$$\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon \boldsymbol{V}^{(2)} + \epsilon^2 \boldsymbol{V}^{(3)} + \cdots = \sum_{n=1}^{\infty} \epsilon^{n-1} \boldsymbol{V}^{(n)} . \tag{4}$$

Here, the coefficient of each terms is taken to be an integral power of some constant ϵ. This infinite summation is rather formal.

By virtue of the recursion relation, the resulting equation is expressed in a compact form:

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} - \epsilon \boldsymbol{X}_s \times \boldsymbol{X}_{ts} + \mathcal{T} \boldsymbol{X}_s , \tag{5}$$

where

$$\mathcal{T} = \frac{1}{2} \epsilon \boldsymbol{X}_t \cdot \boldsymbol{X}_t + C(t) , \tag{6}$$

with $C(t)$ being an arbitrary function of t, and the condition $\boldsymbol{X}_s \cdot \boldsymbol{X}_s = 1$ is to be kept in view. The derivation of (6) is straightforward; we first differentiate the both side of (5) with respect to s, and thereafter take the inner product with $\boldsymbol{X}_s$. Using (5) again, we have $\mathcal{T}_s = \epsilon \boldsymbol{X}_{st} \cdot \boldsymbol{X}_t$, from which (6) follows. Langer and Perline picked out a restricted class of curves, balanced asymptotically linear curves. As a consequence, $C(t)$ is absent in their analysis. We retain it for a wider applicability.

It is illuminating to rewrite (5) into an alternative form. By taking the exterior product with $\boldsymbol{X}_s$, (5) is converted into

$$\boldsymbol{X}_s \times \boldsymbol{X}_t = -\boldsymbol{X}_{ss} + \epsilon \boldsymbol{X}_{st} \,. \tag{7}$$

Then, after changing the variables:

$$\eta = s, \quad \xi = \frac{2}{\epsilon} t + s \,, \tag{8}$$

we arrive at

$$\boldsymbol{X}_{\xi\xi} - \boldsymbol{X}_{\eta\eta} = -\frac{2}{\epsilon} \boldsymbol{X}_\xi \times \boldsymbol{X}_\eta \,. \tag{9}$$

This equation, supplemented by two auxiliary conditions:

$$\boldsymbol{X}_\xi^2 + \boldsymbol{X}_\eta^2 = 1 - \epsilon C(t) \,, \tag{10}$$

$$\boldsymbol{X}_\xi \cdot \boldsymbol{X}_\eta = \frac{\epsilon}{2} C(t) \,, \tag{11}$$

is no other than the Lund-Regge equation [11]. It was derived as a model for the motion of a relativistic string subject to a constant external field. Notice that (10) and (11) differ from the original ones. To gain our expressions, it suffices to choose $x^0 = \frac{1}{2}(\xi + \eta) + \frac{1}{\epsilon} \int^{\frac{\epsilon}{2}(\xi - \eta)} \sqrt{1 - 2\epsilon C(t)} \, dt$ in eq. (3.1) of ref. [11]. Our equation meets the conditions (10) and (11) , which is proved with no difficulty, as follows:

$$\boldsymbol{X}_\xi^2 + \boldsymbol{X}_\eta^2 = \boldsymbol{X}_s^2 - \epsilon \boldsymbol{X}_s \cdot \boldsymbol{X}_t + \frac{\epsilon^2}{2} \boldsymbol{X}_t^2 = 1 - \epsilon C(t) \,, \tag{12}$$

$$\boldsymbol{X}_\xi \cdot \boldsymbol{X}_\eta = \frac{\epsilon}{2} \boldsymbol{X}_t \cdot \left(\boldsymbol{X}_s - \frac{\epsilon}{2} \boldsymbol{X}_t \right) = \frac{\epsilon}{2} C(t) \,. \tag{13}$$

It is noteworthy that the Lund-Regge equation is equivalent to the Lund-Regge-Pohlmeyer-Getmanov equations, a complexified sine-Gordon equation, solvable by the inverse scattering method [11, 12]. Moreover, (9) – (11) ensure not only arclength preservation, but also writhe conservation of an evolving curve.

In keeping with the above procedure of infinite summation, we may deduce the intrinsic form of (5) or (7) by implementing the corresponding infinite summation of elements of the NLS hierarchy. Instead, we directly approach it along the line of Hasimoto's procedure [3] .

For the curve with curvature κ torsion τ, and Frenet-Serret frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$, let us introduce the complex curvature and the complex vector defined

$$\psi = \kappa e^{i\int^s \tau ds}, \quad \boldsymbol{N} = (\boldsymbol{n} + i\boldsymbol{b})e^{i\int^s \tau ds}, \tag{14}$$

From the identities: $\boldsymbol{N} \cdot \boldsymbol{N} = 0$, $\boldsymbol{N} \cdot \boldsymbol{N}^* = 2$, etc., the time derivative of $\boldsymbol{N}$ can be expressed with some real function R and some complex function γ,

$$\boldsymbol{N}_t = iR\boldsymbol{N} + \gamma \boldsymbol{t}. \tag{15}$$

The integrability condition for $\boldsymbol{N}$ leads to

$$\psi_t = -\gamma_s + iR\psi, \tag{16}$$

$$\psi \boldsymbol{t}_t = -iR_s\boldsymbol{N} - \frac{1}{2}\gamma(\psi^*\boldsymbol{N} + \psi\boldsymbol{N}^*). \tag{17}$$

Using (5) and (6), we obtain the expression for R and γ

$$R_s = \frac{1}{2}|\psi|_s^2 - \frac{1}{2}\epsilon|\psi|_t^2 = \boldsymbol{X}_{st} \cdot \boldsymbol{X}_t\,, \tag{18}$$

$$\gamma = -i\psi_s + i\epsilon\psi_t\,, \tag{19}$$

and then the intrinsic equation is

$$\psi_t = i\psi_{ss} + \frac{i}{2}|\psi|^2\psi - i\epsilon\left(\psi_{ts} + \frac{1}{2}\psi\int^s |\psi|_t^2\, ds\right). \tag{20}$$

We remark that the same equation is reached via the use of the recursion operator associated with the NLS hierarchy [4, 5]. According to the form of the operator, for asymptotically linear curve, the indefinite integral in (20) is superseded by a definite integral:

$$\frac{1}{2}\left[\int_{-\infty}^s \kappa\kappa_t\, ds - \int_s^\infty \kappa\kappa_t\, ds\,.\right] \tag{21}$$

Splitting this equation into the real and imaginary part, we have the equations for κ and τ:

$$\kappa_t = -(2\kappa_s\tau + \kappa\tau_s) + \epsilon\left(\kappa_t\tau + \kappa\tau_t + \kappa_s\int^s \tau_t\, ds\right), \tag{22}$$

$$\int^s \tau_t\, ds = \frac{\kappa_{ss}}{\kappa} - \tau^2 + \frac{\kappa^2}{2} - \epsilon\left[\frac{\kappa_{st}}{\kappa} - \tau\int^s \tau_t\, ds + \int^s \kappa\kappa_t\, ds\right], \tag{23}$$

the later of which becomes, after differentiation with respect to s,

$$\tau_t = \left(\frac{\kappa_{ss}}{\kappa}\right)_s - 2\tau\tau_s + \kappa\kappa_s - \epsilon\left[(\frac{\kappa_{st}}{\kappa})_s - \tau_s\int^s \tau_t\, ds - \tau\tau_t + \kappa\kappa_t\right]. \tag{24}$$

The one-soliton solution of (5) and (6) is readily available, by the change of variable, from that of the Lund-Regge equation constructed by Sym *et al.*

[14]. It is a curve of constant torsion τ_0 and is written as

$$X + iY = \frac{2\nu}{\tau_0^2 + \nu^2} \operatorname{sech} Q \, e^{iP} , \tag{25}$$

$$Z = s - \frac{2\nu}{\tau_0^2 + \nu^2} \tanh Q \tag{26}$$

where $\boldsymbol{X} = (X, Y, Z)$ and

$$P = \tau_0 s - \frac{\tau_0^2 - \nu^2 - \epsilon\tau_0(\tau_0^2 + \nu^2)}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} t + c_1 , \tag{27}$$

$$Q = \nu \left[s - \frac{2\tau_0 - \epsilon(\tau_0^2 + \nu^2)}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} t \right] + c_2 , \tag{28}$$

with ν, c_1 and c_2 being arbitrary constants. This has the same form as the Hasimoto soliton of the LIE, except for the dispersion relation. Correspondingly, the one-soliton solution of (22) and (5) is

$$\kappa = 2\nu \operatorname{sech} Q, \quad \tau = \tau_0 . \tag{29}$$

We point out that the coefficients of t in (27) and (28) are obtainable from that of the Hasimoto soliton simply by the replacement $\zeta^2 = (\tau_0 + i\nu)^2 \mapsto \zeta^2/(1 - \epsilon\zeta)$. Further, (25) – (28) are reduced, up to $O(\epsilon)$, to a soliton on a vortex filament with an axial velocity [6, 7, 15].

As a special case, our equation is collapsed into the sine-Gordon equation. In terms of the variables $\hat{t} = t$ and $\hat{s} = s + \frac{1}{\epsilon}t$, (22) and (23) read

$$\kappa_{\hat{t}} + \frac{1}{\epsilon}\kappa_{\hat{s}} = \epsilon\left[\kappa_{\hat{t}}\tau + \kappa\tau_{\hat{t}} + \kappa_{\hat{s}}\int^{\hat{s}}\tau_{\hat{t}}\,d\hat{s}\right], \tag{30}$$

$$\int^{\hat{t}}\tau_{\hat{t}}\,ds + \frac{1}{\epsilon}\tau = -\epsilon\left[\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} - \tau\int^{\hat{s}}\tau_{\hat{t}}\,d\hat{s} + \int^{\hat{s}}\kappa\kappa_{\hat{t}}\,d\hat{s}\right], \tag{31}$$

which leads, by differentiation with respect to $\hat{s}$, to

$$\tau_{\hat{t}} + \frac{1}{\epsilon}\tau_{\hat{s}} = -\epsilon\left[(\frac{\kappa_{\hat{s}\hat{t}}}{\kappa})_{\hat{s}} - \tau_{\hat{s}}\int^{\hat{s}}\tau_{\hat{t}}d\hat{s} - \tau\tau_{\hat{t}} + \kappa\kappa_{\hat{t}}\right]. \tag{32}$$

If we set $\tau = 1/\epsilon$, the first equation is identically satisfied with an appropriate choice of the integral constant such that $\int^{\hat{s}}\tau_{\hat{t}}\,d\hat{s} = 1/\epsilon$. For definiteness, we restrict our attention to the asymptotically linear curves. Their curvature vanishes at infinity. In view of (21), (31) becomes

$$\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} + \frac{1}{2}\left[\int_{-\infty}^{\hat{s}}\kappa\kappa_{\hat{t}}\,d\hat{s} - \int_{\hat{s}}^{\infty}\kappa\kappa_{\hat{t}}\,d\hat{s}\right] = -\frac{1}{\epsilon^3}. \tag{33}$$

Following Nakayama *et al.* [16], we define

$$\theta = \int_{-\infty}^{\hat{s}}\kappa\,d\hat{s}, \tag{34}$$

and prescribe the temporal evolution of κ as

$$\kappa_{\hat{t}} = -\frac{1}{\epsilon^3}\sin\theta. \tag{35}$$

Substituting (34) and (35) and noting from (35) that $\sin\theta \to 0$ as $\hat{s} \to \pm\infty$, we find that (33) holds. The consistency of (34) with (35) gives rise to

$$\theta_{\hat{s}\hat{t}} = -\frac{1}{\epsilon^3}\sin\theta\,. \tag{36}$$

A kink solution of (36) coincides with (29) with $\tau_0 = 1/\epsilon$. Other solutions of (36) include ones that are not covered by soliton solution of (20).

We are grateful to Professor Yukinori Yasui for illuminating discussions on integrable evolution equations.

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