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BEHAVIOR OF RADIALLY SYMMETRIC SOLUTIONS OF A SYSTEM RELATED TO CHEMOTAXIS

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1. Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

$$\begin{cases} b_t = \nabla \cdot (\nabla b - \chi b \nabla \phi(s)) & \text{in} \quad \Omega \times (0,\infty), \\ 0 = \Delta s - s + b & \text{in} \quad \Omega \times (0,\infty), \end{cases} \tag{1.1}$$

under the conditions

$$\begin{cases} \dfrac{\partial b}{\partial n} = \dfrac{\partial s}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ b(\cdot,0) = b_0 \qquad \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where χ is a postive constant and ϕ is a smooth function on $(0,\infty)$ with $\phi' > 0$. The system is a simplified Keller-Segel model. Keller-Segel model was introduced by Keller and Segel [11] to describe the initiation of chemotactic aggregation of cellular slime molds. On Keller-Segel model and simplified Keller-Segel models, time-local existence of the solutions has been studied by [19] and blow-up of the solutions has been studied by [4, 10, 9, 14, 18].

The domain Ω and the non-trivial initial function b_0 are only confined to the following case:

(A1) Ω is the open ball of radius L with center at the origin in $\mathbf{R}^N$.

(A2) b_0 is smooth and nonnegative on $\bar{\Omega}$, and is radially symmetric when $N \geq 2$.

Under these assumptions, there exists a unique solution $(b(x,t), s(x,t))$ to (1.1) and (1.2) defined maximal interval of existence $[0, T_{max})$, which is radially symmetric in x when $N \geq 2$, smooth in $\bar{\Omega} \times (0, T_{max})$ and $b(x,t) > 0$, $s(x,t) > 0$ for $(x,t) \in \Omega \times (0, T_{max})$. If $T_{max} < \infty$,

$$\limsup_{t \to T_{max}} (\|b(\cdot,t)\|_{L^\infty} + \|s(\cdot,t)\|_{L^\infty}) = \infty,$$

by which we mean that $(b(x,t), s(x,t))$ blows up in finite time.

Theorem 1 *Let $N = 1$ and ϕ be smooth on $(0,\infty)$. Then the solution (b,s) to (1.1), (1.2) is globally bounded, that is, $T_{max} = \infty$ and (b,s) satisfies*

$$\sup_{t\geq 0} (\|b(\cdot,t)\|_{L^\infty} + \|s(\cdot,t)\|_{L^\infty}) < \infty.$$

We put

$$M_a(t) = \int_\Omega b(x,t)|x|^a dx \qquad \text{for } 0 \leq t < T_{max},$$

where a is a positive constant. That is called the moment of order a, of $b(\cdot,t)$.

Theorem 2 *Assume $\phi(s) = s^p$ $(p > 0)$, (A1) and (A2).*

(1) $N = 2$:

- (a) *If $0 < p < 1$, then the solution is globally bounded in time.*
- (b) $p = 1$:
  - (i) *If $\|b_0\|_{L^1} < 8\pi/\chi$, then the solution is globally bounded in time.*
  - (ii) *If $\|b_0\|_{L^1} > 8\pi/\chi$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*
- (c) *If $p > 1$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

(2) *If $N \geq 3$ and $M_{(N-2)p+2}(0)$ is sufficiently small, then the solution blows up in finite time.*

Theorem 3 *Assume $\phi(s) = \log s$, (A1) and (A2).*

(1) *If $N = 2$, then the solution is globally bounded in time.*

(2) $N \geq 3$:

- (a) *If $\chi < 2/(N-2)$, then the solution is globally bounded in time.*
- (b) *If $\chi > 2N/(N-2)$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

2. Time-global existence and boundedness

The purpose in this section is to sketch the proofs of Theorem 1 and (i) in Theorems 2 and 3.

Let G be the Green function of $-\Delta + 1$ in Ω with homogeneous Neumann boundary conditions. For $N \geq 2$ we put

$$E(r) = (2\pi)^{-N/2} r^{(2-N)/2} \kappa_{(N-2)/2}(r) \qquad \text{for } r > 0,$$

where κ_ν is the modified Bessel function of the second kind of order ν (see [13]). E is a fundamental solution of $-\Delta + 1$.

For the solution (b,s) to (1.1), (1.2) define the functions S and B by

$$S(r,t) = \int_{|x|\le r} s(x,t)dx, \quad B(r,t) = \int_{|x|\le r} b(x,t)dx \tag{2.1}$$

for $0 \le r \le L$ and $0 \le t < T_{max}$, respectively. B and S satisfy

$$\frac{\partial B}{\partial t} = r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial B}{\partial r}\right) + \frac{\chi}{\omega_N}(B-S)\phi'(s)r^{1-N}\frac{\partial B}{\partial r}, \tag{2.2}$$

$$0 = r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial S}{\partial r}\right) - S + B, \tag{2.3}$$

for $0 < r < L$ and $0 < t < T_{max}$, and

$$B(0,t) = S(0,t) = 0, \quad B(L,t) = S(L,t) = \|b_0\|_{L^1},$$

where ω_N is the surface area of the unit sphere S^{N-1} in $\mathbf{R}^N$.

In order to show the boundedness and time-global existence of solutions (b, s) to (1.1), (1.2) , we begin with the following lemmas. These lemmas are shown by the arguments similar to those in [14] and [18], respectively, so we omit the proofs. In what follows, C denotes a generic positive constant depending on L and N.

Lemma 2. 1 *Let $N \ge 2$. Then*

$$s(x,t) \ge C\|b_0\|_{L^1} \quad \textit{for } x \in \bar{\Omega} \textit{ and } t \in (0, T_{max}).$$

Lemma 2. 2 *If the following condition*

$$\sup_{0\le t<T_{max}} \|s(\cdot,t)\|_{L^\infty} < \infty, \qquad \sup_{0\le t<T_{max}} \|\nabla\phi(s(\cdot,t))\|_{L^\infty} < \infty,$$

holds, then $T_{max} = \infty$ and

$$\sup_{t>0} \|b(\cdot,t)\|_{L^\infty} < \infty.$$

For the following lemma, see [18].

Lemma 2. 3 *Let $N \ge 2$. Then the following holds :*

$$B(|x|,t)E(|x|) \le s(x,t) \le C\|b_0\|_{L^1}E(|x|) \quad \textit{in } \Omega\setminus\{0\}\times(0,T_{max}).$$

Sketch of proofs of Theorem 1 and (i) in Theorems 2 and 3 . By Lemmas 2. 1 , 2. 2 and Appendixes in [14] and [18], it suffices to show that

$$\sup_{0\le t<T_{max}} \|s(\cdot,t)\|_{L^\infty} < \infty, \qquad \sup_{0\le t<T_{max}} \|\nabla s(\cdot,t)\|_{L^\infty} < \infty. \tag{2.4}$$

In the case of $N = 1$, (2.4) is shown by the arguments similar to those in [14]. Hence we will prove (2.4) in the case of $N \ge 2$.

We put

$$\Phi(u) = \begin{cases} p\|b_0\|_{L^1}^p u^{p-1} & \text{in the case of Theorem 2 ,} \\ u^{-1} & \text{in the case of Theorem 3} \end{cases}$$

for $u > 0$. It follows from Lemma 2. 3 , (2.2) and $\partial B/\partial r \geq 0$ that B satisfies

$$\frac{\partial B}{\partial t} \leq r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial B}{\partial r}\right) + \frac{\chi}{\omega_N}\Phi(E)r^{1-N}\frac{\partial B}{\partial r}.$$

We can construct the function $W(r)$ such that

$$W(r) \sim r^N \text{ as } r \to 0,$$

$$\|b_0\|_{L^1} < W(L) \quad \text{and} \quad B(r,0) \leq W(r) \qquad \text{for } 0 \leq r \leq L,$$

and that

$$0 \geq r^{N-1}\frac{d}{dr}\left(r^{1-N}\frac{dW}{dr}\right) + \frac{\chi}{\omega_N}B\phi'(s)r^{1-N}\frac{dW}{dr} \text{ for } 0 < r < L.$$

Hence, the comparison theorem yields that

$$B(r,t) \leq W(r) \qquad \text{for } 0 \leq r \leq L, 0 \leq t < T_{max},$$

which implies $B(r,t) \leq Cr^N$.

Since $B(r,t) \leq Cr^N$ for $0 \leq r \leq L$ and $0 \leq t < T_{max}$, it follows from (2.3) that

$$S(r,t) \leq Cr^N \qquad \text{for } 0 \leq r \leq L, 0 \leq t < T_{max}.$$

Then we have that

$$|\nabla s(x,t)| = \frac{|S(|x|,t) - B(|x|,t)|}{\omega_N |x|^{N-1}} \leq C$$

for $x \in \Omega$ and $0 \leq t < T_{max}$. The boundedness of $\|s(\cdot,t)\|_{L^\infty}$ with respect to $t \in [0, T_{max})$ follows from the estimate above of $|\nabla s|$ and

$$\min_{x\in\bar{\Omega}} s(x,t) \leq \frac{\|b_0\|_{L^1}}{|\Omega|} \qquad \text{for } 0 \leq t < T_{max},$$

where $|\Omega|$ is the volume of Ω. Thus the proofs of (i) of Theorems 2 and 3 are complete.

3. Blow-up of solutions

The purpose in this section is to show the blow-up of solutions for the system (1.1), (1.2) in the case of $N \geq 2$.

In order to show the blow-up of solutions (b,s) to (1.1), (1.2) in [14] and [18], a differential inequality on a moment $M_k(t)$ of b is constructed by use of some estimates of s, and under some conditions on b_0 it is shown that the moment of b converges to 0 as t tends some $T_0 \in (0,\infty)$ by use of the differential inequality.

The following lemma is an immediate consequence of Hölder's inequality.

Lemma 3. 1 *Let f be an integrable function on Ω, and p_1, p_2 and p_3 be numbers satisfying $0 \leq p_1 < p_2 < p_3$. Then*

$$\int_\Omega |f||x|^{p_2}dx \leq \left\{\int_\Omega |f||x|^{p_1}dx\right\}^{(p_3-p_2)/(p_3-p_1)} \left\{\int_\Omega |f||x|^{p_3}dx\right\}^{(p_2-p_1)/(p_3-p_1)}.$$

Let S and B be the same functions as in (2.1). The following lemmas are stated in [15] and [18].

Lemma 3. 2 *The inequality holds :*

$$\begin{aligned}\frac{d}{dt}M_k(t) &\leq k(k+N-2)\int_\Omega b(x,t)|x|^{k-2}dx \\ &+\frac{k\chi}{\omega_N}\int_\Omega \phi'(s(x,t))b(x,t)\left\{S(|x|,t)-B(|x|,t)\right\}|x|^{k-N}dx\end{aligned}$$

on $(0,T_{max})$, *where* $k\geq 2$.

Lemma 3. 3 *Let* $N\geq 3$. *There exists a positive constant* δ *such that*

$$\frac{\partial}{\partial r}\left(r^{N-1}s(x,t)\right)\geq 0 \qquad \text{in } \{x\in R^N : |x|\leq\delta\}\times(0,T_{max}),$$

where $r=|x|$.

Lemma 3. 4 *Let* $N\geq 2$. *Then the following holds :*

$$s(x,t)\leq\frac{1}{\omega_N|x|^{N-1}}\int_{|y|=|x|}E(|x-y|)d\sigma\|b_0\|_{L^1}+\int_\Omega K(x,y)b(y,t)dy$$

in $\Omega\setminus\{0\}\times(0,T_{max})$.

Sketch of proof of (ii) of Theorem 2 . Let $k=(N-2)p+2$. In order to prove the theorem, it suffices to show the following inequality

$$\begin{aligned}\frac{d}{dt}M_k(t) &\leq k(k+N-2)\|b_0\|_{L^1}^{2/k}M_k(t)^{(k-2)/k} \\ &+C\|b_0\|_{L^1}^{p+(k-2)/k}M_k(t)^{2/k}-C\|b_0\|_{L^1}^{p+1}\end{aligned} \tag{3.1}$$

for $t\in(0,T_{max})$. In fact, if $M_k(0)$ is sufficiently small so that the right-hand side of (3.1) is negative at $t=0$, there exists $T_0\in(0,\infty)$ such that

$$M_k(t)\to 0 \quad \text{as} \quad t\to T_0.$$

Hence, T_{max} must be finite and $T_{max}\leq T_0$. By Appendixes in [14] and [18], we have

$$\limsup_{t\to T_{max}}\|b(\cdot,t)\|_{L^\infty}=\infty.$$

Let us first show (3.1) in the case of $p\geq 1$. Using Lemmas 2. 3 and the properties of the fundamental solution, we obtain that

$$\int_\Omega s^{p-1}(x,t)b(x,t)B(|x|,t)|x|^{k-N}dx \;\geq\; C\|b_0\|_{L^1}^{p+1}. \tag{3.2}$$

and that

$$S(|x|,t)\leq C\|b_0\|_{L^1}|x|^2. \tag{3.3}$$

It follows from Lemma 2. 3 and (3.3) and the properties of the fundamental solution that

$$\int_\Omega s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p M_2(t). \tag{3.4}$$

Lemma 3. 2 together with (3.2), (3.4) and Lemma 3. 1 yields (3.1).

Let us consider the case $0 < p < 1$. By Lemmas 2. 3 and the properties of the fundamental solution, we have

$$\begin{aligned}&\int_\Omega s^{p-1}(x,t)b(x,t)B(|x|,t)|x|^{k-N}dx\\ &\ge C\|b_0\|_{L^1}^{p-1}\int_\Omega b(x,t)B(|x|,t)dx = \frac{C}{2}\|b_0\|_{L^1}^{p+1}.\end{aligned} \tag{3.5}$$

It follows from Lemmas 2. 3 and 3. 3 and the properties of the fundamental solution that

$$\int_{|x|\le\delta} s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p\int_{|x|\le\delta} b(x,t)|x|^2dx. \tag{3.6}$$

By Lemma 2. 1 and (3.3), we have

$$\begin{aligned}&\int_{\delta\le|x|\le L} s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx\\ &\le C\|b_0\|_{L^1}^p\int_{\delta\le|x|\le L} b(x,t)|x|^{k-N+2}dx\\ &\le C\|b_0\|_{L^1}^p\delta^{k-N}\int_{\delta\le|x|\le L} b(x,t)|x|^2dx.\end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7) yields that

$$\int_\Omega s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p M_2(t). \tag{3.8}$$

By (3.5) and (3.8), the similar argument to that in the case of $p \ge 1$ gives us (3.1). Thus the proof is complete.

Sketch of proof of (ii) in Theorem 3 . Observe that it follows from Lemma 3. 4 and the properties of the fundermental solution that for $0 \le t < T_{max}$ and $0 < |x| \le L/2$,

$$s(x,t) \;\le\; \left\{\frac{1}{\omega_N(N-2)|x|^{N-2}} + C(|x|^{3-N}+1)\right\}\|b_0\|_{L^1}.$$

For $0 \le t < T_{max}$ and $0 < \delta \le L/2$, we then have that

$$\begin{aligned}&\int_\Omega b(x,t)B(|x|,t)\frac{1}{|x|^{N-2}s(x,t)}dx\\ &\ge \frac{1}{\|b_0\|_{L^1}}\left\{\frac{1}{(N-2)\omega_N} + C\delta\right\}^{-1}\int_{|x|\le\delta} b(x,t)B(|x|,t)dx\end{aligned}$$

$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} B(\delta,t)^2 \tag{3.9}$$

$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} \left(\|b_0\|_{L^1} - \frac{1}{\delta^2} M_2(t)\right)_+^2$$

$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)} \left(\|b_0\|_{L^1} - \frac{2}{\delta^2} M_2(t)\right),$$

where $(\cdot)_+ = \max\{\cdot, 0\}$. It follows from Lemma 3. 3 that

$$\int_\Omega b(x,t) S(|x|,t) \frac{1}{|x|^{N-2} s(x,t)} dx$$

$$= \omega_N \int_{|x|\leq\delta} b(x,t)|x|^2 dx + \int_{\delta\leq|x|\leq L} b(x,t) S(|x|,t) \frac{1}{|x|^{N-2} s(x,t)} dx \tag{3.10}$$

$$\leq CM_2(t)$$

in $(0, T_{max})$. Hence, combining Lemma 3. 2 with (3.9) and (3.10) concludes that

$$\frac{d}{dt} M_2(t) \leq \left\{2N - \frac{(N-2)\chi}{1+C\delta}\right\} \|b_0\|_{L^1} + C\chi(1+\delta^{-2}) M_2(t)$$

on $(0, T_{max})$. Suppose that δ is sufficiently small so that $2N(1+C\delta) - (N-2)\chi < 0$. Using the argument similar to that in the sketch of proof of Theorem 2 , then we have the proof.

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