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Existence and partial regularity for heat flows for a variational functional of degenerate type

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1 Introduction.

Let M, N be compact, smooth orientable Riemannian manifolds of dimension m, l with metrics g, h respectively and suppose that ∂M, $\partial N = \emptyset$. Since N is compact, N may be isometrically embedded into a Euclidean space R^n for some n. For a C^1-map $u : M \to N \subset R^n$, we introduce a variational functional $I(u)$ given by

$$I(u) = \int_M f(|Du|^2)dM, \tag{1.1}$$

where, in local coordinates on M, with $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, $|g| = \det(g_{\alpha\beta})$ and $D_\alpha = \partial/\partial x^\alpha$ $(\alpha = 1, \cdots, m)$,

$$dM = \sqrt{|g|}dx, \quad |Du|^2 = \sum_{\alpha,\beta=1}^{m} \sum_{i=1}^{n} g^{\alpha\beta} D_\alpha u^i D_\beta u^i$$

and f is assumed to satisfy the followings: For simplicity, set $F(\tau) = f(\tau^2)$ for all $\tau \geq 0$. F is a real valued convex C^2- function defined on $[0, +\infty)$ such that $F(0) = F'(0) = 0$ and, with uniform positive constants λ, ω, a, $0 < a < 1$, and $p \geq 2$,

$$\text{(H1)} \quad \lim_{\tau \to 0} F''(\tau)/\tau^{p-2} = \lambda, \tag{1.2}$$

$$\text{(H2)} \quad F''(\tau) - \omega = \mathbf{O}(\tau^{-a})(\tau \to +\infty) \tag{1.3}$$

and, moreover, F is an almost everywhere three times differentiable function in $(0, +\infty)$ satisfying, with a uniform positive constant Λ,

$$\text{(H3)} \quad |F'''(\tau)| \leq \Lambda \tau^{-1} \min\{\tau^{p-2}, 1\} \quad \text{for almost all } \tau > 0. \tag{1.4}$$

The Euler-Lagrange equation of a variational functional I is a degenerate elliptic system of second order partial differential equations

$$-\Delta_M^f u + A^f(u)(Du, Du) = 0, \tag{1.5}$$

where, with the second fundamental form $A(u)(Du, Du)$ of N in R^n,

$$\Delta_M^f u = (\sqrt{|g|})^{-1} D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} f'(|Du|^2) D_\beta u \right),$$

$$A^f(u)(Du, Du) = f'(|Du|^2) g^{\alpha\beta} A(u)(D_\alpha u, D_\beta u).$$

Here and in what follows, the summation notation over repeated indices is adopted.

For $q > 1$, we now define a set of Sobolev mappings between M and N

$$W^{1,q}(M,N) = \{v \in W^{1,q}(M,R^n) : v(x) \in N \text{ a.e. } x \in M\}. \tag{1.6}$$

To investigate the existence and smoothness of critical points of I in $W^{1,2}(M,N)$, which are weak solutions to (1.5), we consider heat flows $u(t) \in W^{1,2}(M,N)$, $0 \leq t < \infty$, for a variational functional (1.1) with a given map $u_0 \in W^{1,2}(M,N)$, where the heat flows are prescribed by a degenerate parabolic system of second order partial differential equations with an initial date

$$\partial_t u - \Delta_M^f u + A^f(u)(Du,Du) = 0 \quad \text{in} \quad (0,\infty)\times M, \tag{1.7 }$$

$$u(0,x) = u_0(x) \qquad \text{for } x \in M. \tag{1.8 }$$

In this paper we report the existence and partial regularity of global weak solutions to (1.7) and (1.8) Previously, Chen and Struwe established the global existence and partial regularity for heat flows for harmonic maps, based on a decay estimate analogous to the monotonicity formula for minimizing harmonic maps (see [2], [15]). We make extension of their results to obtain our desired theorem. Our main aim is to investigate the existence and partial regularity of heat flows for p-harmonic maps. However we are faced with some difficulties, so that we obtained only the subsequent results (refer to [1]).

To state our results, we need some preminalies: Let us denote by $\text{dist}_\delta(z,A)$ and $H^k(\cdot,\delta)$ a distance between a point z and a set A and the k-dimensional Hausdorff measure with respect to a usual parabolic metric δ respectively.

A map $u : [0,+\infty)\times M \to N$ is a global weak solution of (1.7) and (1.8) if and only if $u \in L^\infty((0,+\infty);W^{1,2}(M,N))$ $W^{1,2}((0,+\infty);L^2(M,R^n))$ satisfying, for all $\phi \in L^2((0,+\infty);W^{1,2}(M,\ R^n)) \cap L^\infty((0,+\infty)\times M,R^n)$, the support of which is compactly contained in $(0,+\infty)\times U$ with a coordinate chart U for M,

$$\int_{(0,\infty)\times M} \{\partial_t u\cdot\phi + f'(|Du|^2)g^{\alpha\beta}D_\beta u\cdot D_\alpha\phi + \phi\cdot A^f(u)(Du,Du)\}dMdt = 0 \tag{1.9}$$

and satisfying the initial condition in the sense

$$|u(t)-u_0|_{W^{1,2}(M)} \to 0, \quad t\to 0.$$

Then our main theorem is the following:

Theorem 1 *Suppose $u_0 \in W^{1,2}(M,N)$. Then there exists a global weak solution $u \in L^\infty((0,$ $+\infty);W^{1,2}(M,N)) \cap W^{1,2}((0,+\infty);L^2(M,R^n))$ with the energy inequality*

$$\int_{(0,+\infty)\times M} |\partial_t u|^2 dMdt + \sup_{0\leq t\leq T} I(u(t)) \leq I(u_0). \tag{1.10}$$

Moreover there exist an open set $Q_0 \subset (0,+\infty) \times M$ (with respect to a parabolic metric δ) and a positive number $\alpha, 0 < \alpha < 1$ such that u, Du belong to $C^{0,\alpha}_{loc}(Q_0, \delta)$ and it holds that

$$\partial_t u - \Delta^f_M u + A^f(u)(Du, Du) = 0 \quad \textit{almost everywhere in } Q_0 \tag{1.11}$$

and that

$$H^m((0,+\infty) \times M \setminus Q_0, \delta) < \infty. \tag{1.12}$$

Some standard notations: For $z_0 = (t_0, x_0) \in (0,T) \times R^m$ and $r, \tau > 0$,

$$B_r(x_0) = \{x \in R^m : |x - x_0| < r\}, \tag{1.13}$$
$$Q_{r,\tau}(z_0) = (t_0 - \tau,\ t_0) \times B_r(x_0), \quad P_{r,\tau}(z_0) = (t_0 - \tau, t_0 + \tau) \times B_r(x_0) \tag{1.14}$$

and $Q_r(z_0) = Q_{r,r^2}(z_0)$, $P_r(z_0) = P_{r,r^2}(z_0)$. For vectors $u, v \in R^n$ and $P, Q \in R^{mn}$,

$$u \cdot v = \sum_{i=1}^n u^i v^i, \quad (P,Q) = g^{\alpha\beta} P_\alpha \cdot Q_\beta, \quad |P|^2 = (P,P). \tag{1.15}$$

2 preliminaries.

In this section we gather the estimates for f, without the proofs (refer to [11]). We use the notation: $F(\tau) = f(\tau^2)$ for all $\tau \geq 0$.

First of all we note the followings: The assumption (H1) implies that there exists a positive constant τ_0 depending only on λ such that

$$(\lambda/2)\tau^{p-2} \leq F''(\tau) \leq (3\lambda/2)\tau^{p-2}, \quad 0 \leq \tau \leq \tau_0. \tag{2.1}$$

By the assumption (H2), we are able to choose positive constants L and μ such that

$$0 < \omega - \mu\tau^{-a} \leq F''(\tau) \leq \omega + \mu\tau^{-a}, \quad \tau > \tau_1. \tag{2.2}$$

Lemma 2.1 *For any positive number ρ, there exist positive constants γ_i $(i = 1, \cdots, 4)$ such that F satisfies*

$$\begin{cases} \gamma_1\tau^{p-2} \leq F''(\tau) \leq \gamma_2\tau^{p-2}, & 0 \leq \tau \leq \rho, \\ \gamma_3 \leq F''(\tau) \leq \gamma_4, & \tau > \rho. \end{cases} \tag{2.3}$$

In particular, if $\rho = 1$, then we have

Lemma 2.2 *There exist positive constants γ_i $(i = 1, \cdots, 4)$ such that*

$$\begin{cases} (\gamma_1/2)\tau^{p/2-1} \leq f'(\tau) + 2f''(\tau)\tau \leq (\gamma_2/2)\tau^{p/2-1}, & 0 \leq \tau \leq 1, \\ \gamma_3/2 \leq f'(\tau) + 2f''(\tau)\tau \leq \gamma_4/2, & \tau > 1, \end{cases} \tag{2.4}$$

$$\begin{cases} (\gamma_1/2(p-1))\tau^{p/2-1} \le f'(\tau) \le (\gamma_2/2(p-1))\tau^{p/2-1}, & 0 \le \tau \le 1, \\ (\gamma_1/2(p-1))(1/\sqrt{\tau}) + (\gamma_3/2)(1-1/\sqrt{\tau}) \le f'(\tau) & \\ \quad \le (\gamma_2/2(p-1))(1/\sqrt{\tau}) + (\gamma_4/2)(1-1/\sqrt{\tau}), & \tau > 1 \end{cases} \tag{2.5}$$

and

$$\begin{cases} (\gamma_1/p(p-1))\tau^{p/2} \le f(\tau) \le (\gamma_2/p(p-1))\tau^{p/2}, & 0 \le \tau \le 1, \\ (\gamma_1/p(p-1)) + (\gamma_3/2)(\tau-1) + (\gamma_1/(p-1)-\gamma_3)(\sqrt{\tau}-1) \le f(\tau) & \\ \quad \le (\gamma_2/p(p-1)) + (\gamma_4/2)(\tau-1) + (\gamma_2/(p-1)-\gamma_4)(\sqrt{\tau}-1), & \tau > 1. \end{cases} \tag{2.6}$$

Lemma 2.3 *We are able to choose positive constants γ_{13}, $\bar{\gamma}_{13}$, depending only on γ_1, γ_3, and γ_{24}, depending only on γ_2, γ_4, such that*

$$\gamma_{13}\tau - \bar{\gamma}_{13} \le f(\tau) \le \gamma_{24}\tau \qquad \text{for all } \tau \ge 0. \tag{2.7}$$

For any positive number $\epsilon, 0 < \epsilon < \min\{\gamma_3/2, \gamma_1/p(p-1)\}$, there exists a positive constant τ_2, depending only on ϵ, such that

$$\epsilon\tau < f(\tau) \qquad \text{for all } \tau > \tau_2. \tag{2.8}$$

3 Approximating solutions

In this section we explain the approximate scheme to construct solutions to (1.7) and (1.8).

Since N is smooth and compact, there exists a uniform tubular neighborhood $\mathcal{O}(N) \subset R^n$ of N of width $2\delta_N$ such that each point $p \in \mathcal{O}(N)$ has a unique nearest point $q = \pi_N(p)$ with a distance $\mathrm{dist}(p,N) = |p-q|$ and the projection π_N from $\mathcal{O}(N)$ to N is smooth.

We use a regularization as in [2]. Let χ be a smooth, non-decreasing function such that $\chi(s) = s$ for $s \le \delta_N^2$ and $\chi(s) = 2\delta_N^2$ for $s \ge 4\delta_N^2$. Then the function $\chi(\mathrm{dist}^2(p,N))/2$ is twice differentiable for everywhere $p \in R^n$ and, at points p with $\mathrm{dist}(p,N) \le \delta_N$, its gradient is parallel to $p - \pi_N(p)$ in R^n, that is, it is orthogonal to the tangential space $T_{\pi_N(p)}N$ of N at $\pi_N(p) \in N$.

For a C^1-map v defined on M with a value in R^n, we define the penalized functional with parameters k, $k \to +\infty$:

$$I^k(v) = \int_M \{f(|Du|^2) + Ck\chi(\mathrm{dist}^2(v,N))\}dM, \tag{3.1}$$

where C is a sufficiently large positive constant determined later. We approximate a solution of (0.3) and (0.4) by solutions to gradient flows for the penalized functionals (3.1). For each positive number k, the gradient flows for the penalized functional are prescribed by a system of nonlinear partial differential equations of degenerate parabolic type

$$\partial_t u - \Delta_M^f u + Ck\frac{1}{2}\frac{d}{du}\chi(\mathrm{dist}^2(u,N)) = 0 \quad \text{in } (0,T) \times M \tag{3.2}$$

with an initial data

$$u(0,x) = u_0(x) \quad x \in M. \tag{3.3}$$

We call the equation (3.2) the penalized equation.

We also recall the result for monotone operators (see [[1], Lemma 1.1, Page 27]). Let $X = L^2_{loc}((0,\infty); W^{1,2}(M))$. For any $\phi \in X$, we define $A\phi \in X^*$ by

$$< A\phi, w >= \int_M f'(|D\phi|^2) g^{\alpha\beta} D_\beta\phi \cdot D_\alpha w dM, \quad w \in X. \tag{3.4}$$

It is easy to verify that A defined in (3.4) is hemicontinuous and monotone operator in X. Note that the monotonicity of A follows from the convexity of the functional I.

Then we have, similarly as in [[1], Corollary 1.3, Page 27],

Lemma 3.1 *Let A be a operator defined in (3.4). Assume that $\{u_l\}$ converges to u weakly in $X = L^2_{loc}((0,\infty) : W^{1,2}(M))$ and*

$$\limsup_{l\to\infty} \int_0^T < Au_l, u_l > dt \le -\int_0^T w \cdot Du dt \quad \textit{for any } T > 0, \tag{3.5}$$

where w is the weak limit of the sequence $\{f'(|Du_l|^2)Du_l\} \in L^2_{loc}((0,+\infty); L^2(M))$. Then $\{Au_l\}$ converges to Au weakly in X^*.

We need the result concerning to the compactness of Sobolev embedding (for the proof, refer to [[1], Lemma 1.4, Page 28]).

Lemma 3.2 *Suppose that $\{u_l\}$ is bounded in $L^\infty((0,T); W^{1,q}(M))$, $1 \le q < \infty$, and $\partial_t u_l$ is bounded in $L^2((0,T); L^2(M))$. Then there exist subsequence $\{u_l\}$ and a function $u \in L^\infty((0,T); W^{1,q}(M)) \cap W^{1,2}((0,T); L^2(M))$ such that $\{u_l\}$ converges to u strongly in $L^r((0, T); L^r(M))$ for each $r, q \le r < mq/(m-q)$.*

We now claim the existence of weak solutions to the penalized equation with an initial data $u_0 \in W^{1,2}(M,N)$.

Lemma 3.3 *For every $k \ge 1$, there exists a weak solution for (3.2) and (3.3), satisfying*

$$\begin{aligned} &\int_{(0,T)\times M} |\partial_t u|^2 dM dt + \sup_{0\le t\le T} \{\frac{1}{2}\int_M f(|Du(t)|^2) dM + C\frac{1}{2}k\int_M \chi(dist^2(u(t),N))dM\} \\ &\le \frac{1}{2}\int_M f(|Du_0|^2) dM \quad \textit{for any } t > 0. \end{aligned} \tag{3.6 }$$

Proof. Exploiting Lemma 3.1 with Galerkin's method, we are able to construct weak solutions to the penalized equations. Here we note the monotonicity of the operator A in (3.4) and also use Lemma 3.2 to show the validity of (3.5) for Galerkin approximating solutions.

4 Estimates for a solution to the penalized equation.

In this section we give some estimates for solutions $u = u_k$, $k \geq 1$, to the penalized equations. Now we take a point $(t_0, x_0) \in (0, \infty) \times M$ arbitrarily and fix it. We let a positive constant R_M a lower bound for the injective radius of the exponential map on M such that, for any R, $0 < R < R_M$, the geodesic ball $\mathcal{B}_R(x_0)$ of radius R around x_0 is well-defined and diffeomorphic to the Euclidean ball $B_R(0) \subset R^m$ though the exponential map.
Then we find that, for any $t \in (t_0 - \min\{R_M^2, t_0\}, t_0)$, a map

$$u(t, \exp_{x_0} \cdot) : R^m \ni x \to u(t, \exp_{x_0} x) \in R^n \tag{4.1}$$

is well-defined. We now reset R_M by $\sqrt{\min\{R_M^2, t_0\}}$ and $u(t, x) = u(t, \exp_{x_0} x)$ for any $(t, x) \in (t_0 - R_M^2, t_0) \times B_{R_M}(0) \subset (0, +\infty) \times R^m$ and, moreover, by parallel translation and an appropriate extension of u to $R^m \backslash B_{R_M}(0)$, we regard u as a map defined on $(-R_M^2, 0] \times R^m$ with a value in R^n.

Firstly we state the result concerning to the twice differentiability of solutions (refer to [4], [11]).

Lemma 4.1 *A function* $\min\{|Du|^{p/2-1}, 1\}Du$ *has weak derivatives which lie in* $L^2_{loc}(Q_{R_M})$ *and there exists a positive constant* γ *depending only on* p, M *and* N *such that, for all* $Q_{2r} \subset Q_{R_M}$,

$$\begin{aligned} \sup_{t_0 - r^2 \leq t \leq t_0} \int_{B_r \times \{t\}} |Du|^2 dx + \int_{Q_r} \min\{|Du|^{p-2}, 1\} |D^2 u|^2 dz \\ \leq \gamma r^{-2}(1 + |Du|^2_{L^\infty(Q_{2r})}) \int_{Q_{2r}} |Du|^2 dz + \gamma I(u_0). \end{aligned} \tag{4.2 }$$

We derive a local boundedness of the spatial derivative of u in Q_{R_M}, the proof of which is achieved by Moser's iteration with appropriate regularization (refer to [4], [5]).

Lemma 4.2 *(Local boundedness)* $Du \in L^\infty_{loc}(Q_{R_M})$.

We now claim that Du is locally continuous in Q_{R_M}.

Lemma 4.3 *Du is locally continuous in* Q_R

Proof. We are able to proceed with our consideration similarly as in [5], [6]. Here we may regard our penalized equation as a degenerate parabolic system of p-harmonic type with lower order term of a bounded function and we use the assumption (H3) and the fact that the function $\frac{d^2}{du^2}\chi(\mathrm{dist}^2(u, N))$ is bounded in R^n.

5 Bochner and monotonicity formula.

Let $u = u_k$ be a weak solution to the penalized equation (3.2) and (3.3). Let us take $z_0 = (t_0, x_0) \in (0, +\infty) \times M$ and argue in the same settings as in Sect.4. We recall the settings of u and R_M and use the notation

$$e_k(u) = f(|Du|^2) + Ck\chi(\mathrm{dist}^2(u, N)), \quad \tilde{e}_k(u) = |Du|^2 + k\chi(\mathrm{dist}^2(u, N)). \tag{5.1}$$

We now state the Bochner formula for the energy density $\tilde{e}(u) = \tilde{e}_k(u_k)$ without the proof(refer to [2], [11]). We notice Lemma 4.2 and 4.3. In the estimates, we choose a uniformly and sufficiently large constant C for the technical reason.

Lemma 5.1 *It holds, with a uniform positive constant $\tilde{C}$, for any $\phi \in L^2((-R_M^2, 0); W_0^{1,2}(B_{R_M})) \cap W^{1,2}((-R_M^2, 0); L^2_{loc}(B_{R_M}))$ with $\phi \geq 0$ Q_{R_M} and all t_1, t_2, $-R_M^2 < t_1, t_2 \leq 0$,*

$$\begin{aligned}
&\int_{B_{R_M}\times\{t\}} \tilde{e}(u)\phi dM\Big|_{t=t_1}^{t=t_2} - \int_{(t_1,t_2)\times B_{R_M}} \tilde{e}(u)\partial_t\phi dM dt \\
&+ \int_{(t_1,t_2)\times B_{R_M}} g^{\alpha\beta}(\delta^{\beta\gamma} f'(|Du|^2) + 2f''(|Du|^2) g^{\gamma\bar{\gamma}} D_\beta u \cdot D_{\bar{\gamma}} u) D_\gamma \tilde{e}(u) D_\alpha \phi dz \\
&+ \int_{(t_1,t_2)\times B_{R_M}} 2\phi f'(|Du|^2) g^{\alpha\beta} g^{\gamma\bar{\gamma}} D_\gamma D_\beta u \cdot D_{\bar{\gamma}} D_\alpha u dz \\
&+ \int_{(t_1,t_2)\times B_{R_M}} \phi f''(|Du|^2) g^{\gamma\bar{\gamma}} D_\gamma |Du|^2 D_{\bar{\gamma}} |Du|^2 dz \\
&+ \int_{(t_1,t_2)\times B_{R_M}} \phi \frac{\tilde{C}}{2} k^2 \left|\frac{d}{du}\chi(dist^2(u, N))\right|^2 dM dt \\
&\leq \int_{(t_1,t_2)\times B_{R_M}} \gamma(M, N) \min\{|Du|^{p-2}, 1\} |Du|^2 (1 + |Du|^2) \phi dz.
\end{aligned} \tag{5.2 }$$

The following monotonicity formula is a crucial estimate in our arguments (refer to [2], [15]). Let $\phi \in C_0^\infty(B_{R_M}(0))$ be a cut-off function such that $0 \leq \phi \leq 1$ and $\phi = 1$ in some neighborhood of $B_{R_M}(0)$. Then we introduce, for R, $0 < R < R_M$,

$$\Phi(R, z_0, u) = R^2 \int_{R^m\times\{t=-R^2\}} e_k(u) G_{z_0} \phi^2 \sqrt{|g|} dx$$

and, for $0 < R < R_M/2$,

$$\Psi(R, z_0, u) = \int_{-(2R)^2}^{-R^2} \int_{R^m\times\{t\}} e_k(u) G_{z_0} \phi^2 \sqrt{|g|} dx dt,$$

where, with a positive constant ω in (H2),

$$G_{z_0}(t, x) = (4\pi(-t))^{-m/2} \exp(-|x|^2/2\omega(-t)), \quad -R_M^2 < t < 0.$$

Lemma 5.2 *There exists a positive constant* γ *depending only on* M, N *and* p *such that, for any* $0 < R_0 \le R_1 < R_M$,

$$\Phi(R_0, z_0, u) \le \exp(\gamma(R_1^{1-\epsilon} - R_0^{1-\epsilon}))\Phi(R_1, z_0, u) + \gamma I(u_0)(R_1 - R_0), \tag{5.3}$$

and, for any $0 < R_0 \le R_1 < R_M/2$,

$$\Psi(R_0, z_0, u) \le \exp(\gamma(R_1^{1-\epsilon} - R_0^{1-\epsilon}))\Psi(R_1, z_0, u) + \gamma I(u_0)(R_1 - R_0). \tag{5.4}$$

Proof. We give the proof of (5.3). (5.4) is similarly proven. We proceed with our estimates similarily as in [2], where the restriction (H2) for the growth of F plays an crucial role. For each $0 < R < R_M$, note the following facts. Using a scaling transformation: $(t, x) \to (s, y)$ such that

$$t = R^2 s, \quad x = Ry$$

and setting

$$u_R(s, y) = u(R^2 s, Ry),$$

the equation (3.2) on $(-R_M^2, 0) \times B_{R_M}$ is rewritten as follows: On $(-R_M^2/R^2, 0) \times B_{R_M/R}$

$$\partial_s u_R - \operatorname{div}(f'(R^{-2}|Du_R|^2)Du_R) + R^2 C\frac{k}{2}\frac{d}{du}\chi(\operatorname{dist}^2(u_R, N)) = 0. \tag{5.5}$$

Also note that

$$\begin{aligned}\Phi(R, z_0, u) &= R^2 \int_{R^m \times \{t = t_0 - R^2\}} e_k(u) G_{z_0} \phi^2 \sqrt{|g|} dx \\ &= R^2 \int_{R^m \times \{s = -1\}} \{f(R^{-2} g^{\alpha\beta}(R\cdot) D_\alpha u_R \cdot D_\beta u_R) \\ &\qquad + Ck\chi(\operatorname{dist}^2(u_R, N))\} G_0 \phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy. \end{aligned} \tag{5.6 }$$

We now calculate $\frac{d}{dR}\Phi(R, z_0, u)|_R$ for any $0 < R < r_M$. Set $G = G_0$. We demonstrate only formal calculations, the justification of which is made in [11].

$$\begin{aligned}&\frac{d}{dR}\Phi(R, z_0, u)|_R \\ &= 2R \int_{R^m \times \{s=-1\}} \{f(R^{-2}|Du_R|^2) + Ck\chi(\operatorname{dist}^2(u_R, N))\} G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy \\ &\quad + R^2 \int_{R^m \times \{s=-1\}} \Big\{ f'(R^{-2}|Du_R|^2)\frac{d}{dR}(R^{-2}|Du_R|^2) \\ &\qquad + Ck\frac{d}{dR}u_R \cdot \frac{d}{du}\chi(\operatorname{dist}^2(u_R, N))\Big\} G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy \\ &= 2R \int_{R^m \times \{s=-1\}} \{f(R^{-2}|Du_R|^2) - R^{-2}|Du_R|^2 f'(R^{-2}|Du_R|^2)\} G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy\end{aligned}$$

$$
\begin{aligned}
&+2R\int_{R^m\times\{s=-1\}} Ck\chi(\mathrm{dist}^2(u_R,N))G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy\\
&+2\int_{R^m\times\{s=-1\}}\Big\{g^{\alpha\beta}D_\alpha\frac{d}{dR}u_R\cdot D_\alpha u_R f'(R^{-2}|Du_R|^2)\\
&\qquad +Ck\frac{d}{dR}u_R\cdot\frac{d}{du}\chi(\mathrm{dist}^2(u_R,N))\Big\}G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy\\
&+\int_{R^m\times\{s=-1\}} y\cdot Dg^{\alpha\beta}D_\alpha u_R\cdot D_\alpha u_R f'(R^{-2}|Du_R|^2)G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy\\
&+2\int_{R^m\times\{s=-1\}} e_k(u)G\phi(R\cdot)y\cdot D\phi(R\cdot)\sqrt{|g|}(R\cdot)dy\\
&+\int_{R^m\times\{s=-1\}} e_k(u)G\phi^2(R\cdot)\frac{y\cdot D|g|}{|g|}\sqrt{|g|}(R\cdot)dy\\
&= I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
\tag{5.7}
$$

We now make an estimation of I_1. Split the integrations into four parts:

$$
\begin{aligned}
I_1 &= 2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tilde{\tau}_0\}} f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\
&\quad +2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tilde{\tau}_0\}} f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\
&\quad -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tilde{\tau}_0\}} R^{-2}|Du_R|^2f'(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\
&\quad -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tilde{\tau}_0\}} R^{-2}|Du_R|^2f'(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\
&= I_{11}+I_{12}+I_{21}+I_{22}.
\end{aligned}
\tag{5.8}
$$

where $\tilde{\tau}_0$ is a positive constant determined later. We know that

$$
\begin{aligned}
I_{11} &\ge 0,\\
I_{21} &= -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tilde{\tau}_0\}} R^{-2}|Du_R|^2f'(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\
&\ge -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tilde{\tau}_0\}} \tilde{\tau}_0\Big(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_4}{2}\Big)G\phi^2\sqrt{|g|}dy.
\end{aligned}
\tag{5.9}
$$

To estimate $I_{12}+I_{22}$, we note, by simple calculation,

$$
f(\tau^2)-\tau^2f'(\tau^2)=f(\tau_0^2)+f(\tau^2)-f(\tau_0^2)-\tau(\tau f'(\tau^2)-\tau_0f'(\tau_0^2)+\tau_0f'(\tau_0^2)). \tag{5.10}
$$

Noting that F is of $C^2([0,+\infty))$, we estimate as follows:

$$
\begin{aligned}
f(\tau^2)-f(\tau_0^2) &= \int_{\tau_0}^{\tau} 2sf'(s^2)ds\\
&= 2\int_{\tau_0}^{\tau}(sf'(s^2)-\tau_0f'(\tau_0^2))ds+2\tau_0f'(\tau_0^2)(\tau-\tau_0)
\end{aligned}
$$

$$= 2\int_{\tau_0}^{\tau}\int_{\tau_0}^{s}(f'(\xi^2)+2\xi^2 f''(\xi^2)-\omega/2)d\xi ds$$
$$+(\omega/2)(\tau-\tau_0)^2+2\tau_0 f'(\tau_0^2)(\tau-\tau_0),$$
$$\tau f'(\tau^2)-\tau_0 f'(\tau_0^2) = \int_{\tau_0}^{\tau}(f'(s^2)+2sf''(s^2)-\omega/2)ds+(\omega/2)(\tau-\tau_0). \quad (5.11\ )$$

Applying (2.2) to the above inequalities and substituting the resulting inequalities into (5.10), we have, for any $\tau > \tau_0$,

$$|f(\tau^2)-\tau^2 f'(\tau^2)| \le f(\tau_0^2)+2\tau_0^2 f'(\tau_0^2)+\tau_0(f'(\tau_0^2)+\omega/2)\tau+\mu(4-a)\tau^{2-a}/2(1-a)(2-a). \quad (5.12)$$

Now we put, for any positive number ϵ_1 and δ,

$$\tilde{\tau_0} = \max\{\tau_0^2, \tau_2(\epsilon_1), (\epsilon_1 R^\delta)^{-2/a}\}. \quad (5.13)$$

Then we obtain from (5.12) and (2.8) that, for all $\tau \ge \sqrt{\tilde{\tau_0}}$,

$$|f(\tau^2)-\tau^2 f'(\tau^2)| \le f(\tau_0^2)+2\tau_0^2 f'(\tau_0^2)+\tau_0(f'(\tau_0^2)+\omega/2)\tau+\mu(4-a)R^\delta f(\tau^2)/2(1-a)(2-a). \quad (5.14)$$

Thus we have

$$I_{12}+I_{22} \ge -2(f(\tau_0^2)+2\tau_0^2 f'(\tau_0^2))R\int_{R^m\times\{s=-1\}} G\phi^2\sqrt{|g|}dy$$
$$-2\tau_0(f'(\tau_0^2)+\omega/2)R\int_{R^m\times\{s=-1\}} R^{-1}|Du_R|G\phi^2\sqrt{|g|}dy$$
$$-\frac{\mu(3-a)}{(1-a)(2-a)}R^{\delta-1}R^2\int_{R^m\times\{s=-1\}} f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy. \quad (5.15\ )$$

Estimating the second term in the right hand of the above by Young's inequality and (2.7), we have

$$I_{12}+I_{22} \ge -2(f(\tau_0^2)+2\tau_0^2 f'(\tau_0^2))R\int_{R^m\times\{s=-1\}} G\phi^2\sqrt{|g|}dy$$
$$-\tau_0^2(f'(\tau_0^2)+\omega/2)^2\int_{R^m\times\{s=-1\}} G\phi^2\sqrt{|g|}dy$$
$$-R^2\int_{R^m\times\{s=-1\}}(\gamma_{13})^{-1}(\bar{\gamma}_{13}+f(R^{-2}|Du_R|^2))G\phi^2\sqrt{|g|}dy$$
$$-\frac{\mu(3-a)}{(1-a)(2-a)}R^{\delta-1}R^2\int_{R^m\times\{s=-1\}} f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy \quad (5.16\ )$$

Substituting (5.9) and (5.16) into (5.8), we have

$$I_1 \ge -\tau_0^2(f'(\tau_0^2)+\omega/2)^2\int_{R^m\times\{s=-1\}} G\phi^2\sqrt{|g|}dy$$
$$-2R\Big\{f(\tau_0^2)+2\tau_0^2 f'(\tau_0^2)+\tilde{\tau_0}\Big(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_3}{2}\Big)\Big\}\int_{R^m\times\{s=-1\}} G\phi^2\sqrt{|g|}dy$$

$$-R^2 \int_{R^m\times\{s=-1\}} (\gamma_{13})^{-1}(f(R^{-2}|Du_R|^2)+\bar{\gamma}_{13})G\phi^2\sqrt{|g|}dy$$
$$-\frac{\mu(3-a)}{(1-a)(2-a)}R^{\delta-1}R^2\int_{R^m\times\{s=-1\}} f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy. \qquad (5.17\ )$$

We now treat I_3 in (5.7). Using (5.5) and $D_\alpha G = -y^\alpha G/\omega(-s)$, we have, by integration by parts,

$$\begin{aligned}
I_3 &= 2\int_{R^m\times\{s=-1\}} \frac{d}{dR}u_R\cdot\Big\{-\Delta_M^f u_R + C\frac{k}{2}\frac{d}{du}\chi(\mathrm{dist}^2(u_R,N))\Big\}G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)dy\\
&\quad -2\int_{R^m\times\{s=-1\}} f'(R^{-2}|Du_R|^2)g^{\alpha\beta}\frac{d}{dR}u_R\cdot D_\beta u_R D_\alpha(G\phi^2(R\cdot))\sqrt{|g|}dy\\
&= -2\int_{R^m\times\{s=-1\}} \frac{d}{dR}u_R\cdot\partial_s u_R G\phi^2\sqrt{|g|}dy\\
&\quad +\int_{R^m\times\{s=-1\}} f'(R^{-2}|Du_R|^2)g^{\alpha\beta}\frac{d}{dR}u_R\cdot D_\beta u_R\Big(\frac{y^\alpha}{(\omega/2)(-s)}\Big)G\phi^2\sqrt{|g|}dy\\
&\quad +2R\int_{R^m\times\{s=-1\}} f'(R^{-2}|Du_R|^2)g^{\alpha\beta}\frac{d}{dR}u_R\cdot D_\beta u_R G\phi(D_\alpha\phi)\sqrt{|g|}dy\\
&= R^{-1}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}|2s\partial_s u_R + y^\alpha D_\alpha u_R|^2 G\phi^2\sqrt{|g|}dy\\
&\quad -R^{-1}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}g^{\alpha\beta}y^\alpha D_\beta u_R(2s\partial_s u_R + y\cdot Du_R)\cdot(y\cdot Du_R)\\
&\quad\quad \times(1-2f'(R^{-2}|Du_R|^2)/\omega)G\phi^2\sqrt{|g|}dy\\
&\quad +R^{-1}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}(g^{\alpha\beta}-\delta^{\alpha\beta})y^\alpha D_\beta u_R\\
&\quad\quad \cdot(2s\partial_s u_R + y\cdot Du_R)G\phi^2\sqrt{|g|}dy.
\end{aligned} \qquad (5.18\ )$$

The latter is bounded from below by

$$\begin{aligned}
&\frac{R^{-1}}{2}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}|2s\partial_s u_R + y^\alpha D_\alpha u_R|^2 G\phi^2\sqrt{|g|}dy\\
&-R^{-1}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}|1-f'(R^{-2}|Du_R|^2)/(\omega/2)|^2|y\cdot Du_R|^2 G\phi^2\sqrt{|g|}dy\\
&-R^{-1}\int_{R^m\times\{s=-1\}} \frac{1}{(-s)}|(g^{\alpha\beta})-(\delta^{\alpha\beta})|^2|y|^2|Du_R|^2 G\phi^2\sqrt{|g|}dy.
\end{aligned} \qquad (5.19\ )$$

For the purpose of an evaluation of the second term in (5.19), take positive numbers δ and ϵ_2. Then, by (2.5), for any $\tau_3 \geq ((\epsilon_2)^{1/2}R^\delta)^{-2/a}$,

$$\begin{aligned}
&R^{-1}\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\leq\tau_3\}} \frac{1}{(-s)}|1-f'(R^{-2}|Du_R|^2)/(\omega/2)|^2|y\cdot Du_R|^2 G\phi^2\sqrt{|g|}dy\\
&\leq \Big\{1+\frac{2}{\omega}\Big(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_4}{2}\Big)\Big\}^2\tau_3 R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\leq\tau_3\}} |y|^2 G\phi^2\sqrt{|g|}dy.
\end{aligned} \qquad (5.20\ )$$

To estimate the remainder of the second term in (5.19), we note the following estimates for all $\tau > \tau_0$:

$$\begin{aligned}
&(\omega/2 - f'(\tau^2))\tau \\
&= (\omega/2)(\tau - \tau_0) - (f'(\tau^2)\tau - f'(\tau_0^2)\tau_0) + (\omega/2 - f'(\tau_0^2))\tau_0 \\
&= \int_{\tau_0}^{\tau} (\omega/2 - (f'(s^2) + 2s^2 f''(s^2)))ds + (\omega/2 - f'(\tau_0^2))\tau_0,
\end{aligned} \tag{5.21}$$

so that, applying (2.2) for the first term in the right hand, we have

$$\begin{aligned}
&|(\omega/2 - f'(\tau^2))\tau| \\
&\leq \int_{\tau_0}^{\tau} |\omega/2 - (f'(s^2) + 2s^2 f''(s^2))|ds + |\omega/2 - f'(\tau_0^2)|\tau_0 \\
&\leq (\mu/2(1-a))\tau^{1-a} + |\omega/2 - f'(\tau_0^2)|\tau_0.
\end{aligned} \tag{5.22}$$

From (5.22) we obtain, for any $\tau_3 \geq \max\{\tau_0^2, ((\epsilon_2)^{1/2} R^\delta)^{-2/a}\}$,

$$\begin{aligned}
&R^{-1} \int_{R^m \times \{s=-1\} \cap \{R^{-2}|Du_R|^2 > \tau_3\}} \frac{1}{(-s)} |1 - f'(R^{-2}|Du_R|^2)/(\omega/2)|^2 |y \cdot Du_R|^2 G\phi^2 \sqrt{|g|} dy \\
&\leq \frac{4R}{\omega^2} \int_{R^m \times \{s=-1\} \cap \{R^{-2}|Du_R|^2 > \tau_3\}} |\omega/2 - f'(R^{-2}|Du_R|^2)|^2 R^{-2}|Du_R|^2 |y|^2 G\phi^2 \sqrt{|g|} dy \\
&\leq \frac{4}{\omega^2} |\omega/2 - f'(\tau_0^2)|^2 \tau_0^2 R \int_{R^m \times \{s=-1\} \cap \{R^{-2}|Du_R|^2 > \tau_3\}} \frac{1}{(-s)} |y|^2 G\phi^2 \sqrt{|g|} dy \\
&+ \frac{\mu^2 R^{-1+2\delta} R^2}{\omega^2 (1-a)^2} \int_{R^m \times \{s=-1\} \cap \{R^{-2}|Du_R|^2 > \tau_3\}} \frac{1}{(-s)} \epsilon_2 R^{-2} |Du_R|^2 |y|^2 G\phi^2 \sqrt{|g|} dy.
\end{aligned} \tag{5.23}$$

Similarly as in (5.14), for a positive number ϵ_2, $0 < \epsilon_2 < \gamma_3/2$, we are able to choose a positive number $\tau_3 = \max\{\tau_0^2, \tau_2(\epsilon_2), ((\epsilon_2)^{1/2} R^\delta)^{-2/a}\}$ such that the latter is bounded from above by

$$R^{-1+2\delta} R^2 \int_{R^m \times \{s=-1\}} \frac{1}{(-s)} f(R^{-2}|Du_R|^2) |y|^2 G\phi^2 \sqrt{|g|} dy. \tag{5.24}$$

We also make estimation of the last term in (5.19)

$$\begin{aligned}
&R^{-1} \int_{R^m \times \{s=-1\}} \frac{1}{(-s)} |(g^{\alpha\beta}) - (\delta^{\alpha\beta})|^2 |y|^2 |Du_R|^2 G\phi^2 \sqrt{|g|} dy \\
&\leq \gamma(M) R^3 \int_{R^m \times \{s=-1\}} \frac{1}{(-s)} (R^{-2}|Du_R|^2) |y|^4 G\phi^2 \sqrt{|g|} dy \\
&\leq \gamma(M) R^3 \int_{R^m \times \{s=-1\}} \frac{1}{(-s)} \frac{f(R^{-2}|Du_R|^2) + \bar{\gamma}_{13}}{\gamma_{13}} |y|^4 G\phi^2 \sqrt{|g|} dy,
\end{aligned} \tag{5.25}$$

where we used (2.7) and the estimate with a positive constant $\gamma(M)$ depending only on M

$$\begin{aligned}
|(g^{\alpha\beta})(Ry) - (\delta^{\alpha\beta})| &= |(g^{\alpha\beta})(Ry) - (g^{\alpha\beta})(0)| \\
&\leq \gamma(M) R|y| \quad \text{for any } x = Ry \in B_{R_M}(0).
\end{aligned} \tag{5.26}$$

We are able to proceed to the estimates for (5.23) and (5.25) as follows: for a positive number $\bar{\delta}$ which is determined later,

$$\begin{aligned}&\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2G\phi^2\sqrt{|g|}dy\\&=\int_{R^m\times\{s=-1\}\cap\{|y|\le R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2G\phi^2\sqrt{|g|}dy\\&+\int_{R^m\times\{s=-1\}\cap\{|y|>R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2G\phi^2\sqrt{|g|}dy.\end{aligned}\tag{5.27 }$$

The first term in (5.27) is estimated from above by

$$R^{-2\bar{\delta}}\int_{R^m\times\{s=-1\}\cap\{|y|\le R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy.$$

Noting that, if $|y|>R^{-\bar{\delta}}$,

$$|y|^2G\le\gamma(m)\exp\{-R^{-2\bar{\delta}}/16\gamma_{34}\}$$

and exploiting our energy inequality (3.6), we have, for the second term in (5.27),

$$\begin{aligned}&\int_{R^m\times\{s=-1\}\cap\{|y|>R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2G\phi^2\sqrt{|g|}dy\\&\le\gamma(m)\exp\{-R^{-2\bar{\delta}}/16\gamma_{34}\}R^{-m}\int_{B_{R_M}\times\{t=-R^2\}}f(|Du|^2)\sqrt{|g|}dx\\&\le\gamma(m)\int_M f(|Du_0|^2)dM.\end{aligned}\tag{5.28 }$$

Substituting (5.27) and (5.28) into (5.23) and (5.25) and combining the resulting inequality with (5.20), we obtain from (5.18) and (5.19) the estimate for I_3 in (5.7): For a positive number $\tau_3=\max\{\ \tau_0^2,\tau_2(\epsilon_2),\ (\epsilon_2^{1/2}R)^{-2/a}\}$

$$\begin{aligned}I_3\ge&-\frac{1}{2}\Big\{1+\frac{2}{\omega}\Big(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_4}{2}\Big)\Big\}^2\tau_3R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tau_3\}}|y|^2G\phi^2\sqrt{|g|}dy\\&-\frac{2}{\omega^2}|\omega/2-f'(\tau_0^2)|^2\tau_0^2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tau_3\}}\frac{1}{(-s)}|y|^2G\phi^2\sqrt{|g|}dy\\&-\frac{\mu^2}{2\omega^2(1-a)^2}R^{-1+2\delta}R^2R^{-2\bar{\delta}}\int_{R^m\times\{s=-1\}\cap\{|y|\le R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)G\phi^2\sqrt{|g|}dy\\&-\gamma(m)R^{-1+2\delta}R^2\int_M f(|Du_0|^2)dM.\end{aligned}\tag{5.29 }$$

We now treat the remaining error terms in (5.7). Similarly as in (5.25), by (2.7), we have

$$|I_4|\ \le\ \gamma(M)R^2\int_{R^m\times\{s=-1\}}(f(R^{-2}|Du_R|^2)+\bar{\gamma}_{13})/\gamma_{13}|y|G\phi^2\sqrt{|g|}dy$$

$$\leq \gamma R^{-\delta} R^2 \int_{R^m \times \{s=-1\}} f(R^{-2}|Du_R|^2 G\phi^2 \sqrt{|g|}dy$$
$$+\gamma I(u_0) + \gamma R^2 \int_{R^m \times \{s=-1\}} |y| G\phi^2 \sqrt{|g|}dy. \tag{5.30}$$

Using Young's inequality, we have, similarly as in (5.27) and (5.28),

$$|I_5| \leq \frac{1}{2}R^2 \int_{R^m \times \{s=-1\}} e_k(u) G\phi^2 \frac{y \cdot D|g|}{|g|} \sqrt{|g|}dy$$
$$+\frac{1}{2}R^2 \int_{R^m \times \{s=-1\}} e_k(u) G|y \cdot D\phi|^2 \sqrt{|g|}dy$$
$$\leq \gamma R^2 \int_{R^m \times \{s=-1\}} e_k(u) G\phi^2 \sqrt{|g|}dy + \gamma(M) I(u_0). \tag{5.31}$$

Simply we have

$$|I_6| \leq \gamma R^2 \int_{R^m \times \{s=-1\}} e_k(u) G\phi^2 \sqrt{|g|}\phi^2 \sqrt{|g|}dy. \tag{5.32}$$

Gathering the estimates (5.17) and (5.29)-(5.32) with (5.7), we obtain

$$\frac{d}{dR}\Phi(R, z_0, u)$$
$$\geq -\gamma(\mu, a, \gamma_{13}, \bar{\gamma}_{13})(1 + R^{\delta-1})\Phi(R, z_0, u) - \gamma(\omega, \mu, a) R^{-1+2\delta-2\bar{\delta}}\Phi(R, z_0, u)$$
$$-R^{1+2\delta}\gamma(m) I(u_0) - \gamma(\omega, \tau_0, p, \gamma_2, \gamma_3)(1 + \tilde{\tau}_0 R) \int_{R^m \times \{s=-1\}} G\phi^2 \sqrt{|g|}dy$$
$$-\gamma(\omega, \tau_0, \gamma_2, \gamma_4)(1 + \tau_3 R) \int_{R^m \times \{s=-1\}} |y|^2 G\phi^2 \sqrt{|g|}dy, \tag{5.33}$$

from which the desired estimate follows, if, recalling (5.13) and (5.24) for $\tilde{\tau}_0$ and τ_3 respectively, δ and $\bar{\delta}$ are taken so small.

Now we derive a-priori estimates for a weak solution u to (3.2) and (3.3). We note Lemmata 4.2 and 4.3. We recall the notation: $\tilde{e}_k(u) = |Du_k|^2 + k\chi(\mathrm{dist}^2(u_k, N))$.

Lemma 5.3 *($\epsilon-$regularity theorem) There exist positive constants ϵ_0 and R_0, $0 < R_0 < \min\{R_M, 1\}$, depending only on $I(u_0), \gamma_1, \gamma_4, \omega, \bar{\gamma}_{13}, N, m$ and p such that, for any weak solution u to (3.2) and (3.3), the following holds: If, for some $t_0 > 0$ and R, $0 < R < \min\{R_0, (t_0/4)^{1/2}\}$, there holds*

$$\Psi(R, z_0, u) = \int_{t_0-(2R)^2}^{t_0-R^2} \int_{R^m \times \{t\}} e_k(u) G_{(t_0,x_0)}\phi^2 \sqrt{|g|}dxdt < \epsilon_0, \tag{5.34}$$

then

$$\sup_{Q_{R/8}(t_0,x_0)} \tilde{e}_k(u) \leq \gamma(R/8)^{-2} \tag{5.35}$$

with a uniform positive constant γ.

Proof. We proceed with our investigations similarly as in [2],[15]. Set $r_1 = \delta R$ with δ, $0 < \delta < 1/2$. Arguing with our monotonicity formula (5.3) and (5.4) and the smallness condition (5.34) (refer to [2]), we have, for positive numbers r, σ, $0 < r, \sigma < r_1$ and $r+\sigma < r_1$, and $z_0 \in P_r$,

$$\sigma^{-m} \int_{Q_\sigma(z_0)} e_k(u)\sqrt{|g|}dxdt \le \epsilon. \tag{5.36}$$

Since $u \in C^0_{loc}((-R^2_M, 0); C^1_{loc}(B_{R_M}(0))$, there exists $\sigma_0, 0 \le \sigma_0 < r_1$, such that

$$(r_1 - \sigma_0)^2 \sup_{Q_{\sigma_0}} \tilde{e}_k(u) = \max_{0 \le \sigma \le r_1} \{(r_1 - \sigma)^2 \sup_{Q_\sigma} \tilde{e}_k(u)\}. \tag{5.37}$$

Here, if $\sigma_0 = r_1$, the desired estimate (3.2) immediately follows. Moreover, we find that there exists $(t_0, x_0) \in \overline{Q_{\sigma_0}}$ such that

$$\sup_{Q_{\sigma_0}} \tilde{e}_k(u) = \tilde{e}_k(u)(t_0, x_0).$$

Now set $\tilde{e}_0 = \tilde{e}_k(u)(t_0, x_0)$ and $\rho_0 = (1/2)(r_1 - \sigma_0)$. By choice of σ_0 and (t_0, x_0)

$$\sup_{Q_{\rho_0}(t_0,x_0)} \tilde{e}_k(u) \le \sup_{Q_{\sigma_0+\rho_0}} \tilde{e}_k(u) \le 4\tilde{e}_0. \tag{5.38}$$

Introduce

$$\begin{aligned} r_0 &= \rho_0\sqrt{\tilde{e}_0/2}, \\ v(s,y) &= u(t_0 + s/(\tilde{e}_0/2), x_0 + y/\sqrt{\tilde{e}_0/2}). \end{aligned} \tag{5.39 }$$

We now show that $r_0 \le 1$. First note that, by (3.2) in Q_{ρ_0} v satisfies, almost everywhere in Q_{r_0},

$$\partial_s v - \operatorname{div}(f'((\tilde{e}_0/2)|Dv|^2)Dv) + \frac{C}{2}(k/(\tilde{e}_0/2))\frac{d}{dv}\chi(\operatorname{dist}^2(v, N)) = 0. \tag{5.40}$$

Moreover (5.39) and (5.38) imply that, with $\tilde{k} = k/(\tilde{e}_0/2)$,

$$\tilde{e}_{\tilde{k}}(v)(0,0) = 2, \quad \sup_{Q_{r_0}} \tilde{e}_{\tilde{k}}(v) \le 8. \tag{5.41}$$

Similarly as in the proof of Lemma 5.2 , we have Bochner type estimate for $\tilde{e}_{\tilde{k}}(v)$. For simplicity we put $\tilde{e}(v) = \tilde{e}_{\tilde{k}}(v)$. Set $B = B_{r_0}$. v satisfies, for $\phi \in L^2((-(r_0)^2, 0); W^{1,2}_0(B_{r_0}))$ $\cap W^{1,2}$ $((-(r_0)^2, 0); L^2(B_{r_0}))$ with $\phi \ge 0$ in Q_{r_0} and all intervals $(t_1, t_2) \subset (-(r_0)^2, 0)$,

$$\begin{aligned} &\int_{R^m \times \{t\}} \tilde{e}(v)\phi dx\Big|_{t=t_1}^{t=t_2} - \int_{R^m \times (t_1,t_2)} \tilde{e}(v)\partial_t\phi dM dt \\ &+ \int_{R^m \times (t_1,t_2)} g^{\alpha\beta}(\delta^{\beta\gamma} f'((\tilde{e}_0/2)|Du|^2) \end{aligned}$$

$$
\begin{aligned}
&+2(\tilde{e}_0/2)f''((\tilde{e}_0/2)|Du|^2)g^{\gamma\bar{\gamma}}D_\beta u\cdot D_{\bar{\gamma}}u)D_\gamma\tilde{e}(v)D_\alpha\phi dz\\
&+\int_{R^m\times(t_1,t_2)}2f'((\tilde{e}_0/2)|Du|^2)g^{\alpha\beta}g^{\gamma\bar{\gamma}}D_\gamma D_\beta u D_{\bar{\gamma}}D_\alpha u\phi dz\\
&+\int_{R^m\times(t_1,t_2)}(\tilde{e}_0/2)f''((\tilde{e}_0/2)|Du|^2)g^{\gamma\bar{\gamma}}D_\gamma|Du|^2D_{\bar{\gamma}}|Du|^2\phi dz\\
&+\int_{Q_{2r}}\phi\frac{\tilde{C}}{2}\Big(\frac{k}{(\tilde{e}_0/2)}\Big)^2\Big|\frac{d}{du}\chi(\mathrm{dist}^2(u,N))\Big|^2dMdt\\
\le&\int_{R^m\times(t_1,t_2)}\gamma(M,N)\min\{(\sqrt{\tilde{e}_0/2}|Du|)^{p-2},1\}|Dv|^2((2/\tilde{e}_0)+|Dv|^2)\phi dz. \quad (5.42\ )
\end{aligned}
$$

Now we assume that $r_0>1$. Then, we are able to derive Harnack type estimate from (5.42) (see [11] for the proof).

Lemma 5.4 *(Harnack estimate) There exists a positive constant γ depending only on γ_1,γ_4, M,N and M such that one of the following inequalities hold, either*

$$\sup_{Q_{1/2}}\tilde{e}(v)\le 1 \tag{5.43}$$

or

$$\sup_{Q_{1/2}}\tilde{e}(v)\le\gamma\Big(\frac{1}{|Q_1|}\int_{Q_1}\tilde{e}(v)^2dMdt\Big)^{1/2}. \tag{5.44}$$

If (5.43) holds, then, by scaling back, we have

$$
\begin{aligned}
2=\tilde{e}_k(u)(t_0,x_0)/(\tilde{e}_0/2)&=\tilde{e}(v)(0,0)\\
&\le\sup_{Q_{1/2}}\tilde{e}(v)\le 1, \qquad (5.45\ )
\end{aligned}
$$

which gives the contradiction.

Otherwise, that is, (5.44) holds. Noting that $r_0>1$ implies $\sigma_0+1/\sqrt{\tilde{e}_0/2}\le\sigma_0+\rho_0<r_1$ and adapting (5.36) with $\sigma=1/\sqrt{\tilde{e}_0/2}$, we have, by scaling back,

$$
\begin{aligned}
&\int_{Q_1}\tilde{e}(v)^2dz\le 8\int_{Q_1}\tilde{e}(v)^2dz=8(\sqrt{\tilde{e}_0/2})^m\int_{Q_{1/\sqrt{\tilde{e}_0/2}(t_0,x_0)}}\tilde{e}_k(u)dz\\
&\le 8(\max\{\gamma_{13},1\}/\gamma_{13})(\sqrt{\tilde{e}_0/2})^m\int_{Q_{1/\sqrt{\tilde{e}_0/2}(t_0,x_0)}}\tilde{e}_k(u)dz+8(\bar{\gamma}_{13}/\gamma_{13})\omega_m(\delta R/2)^2\\
&\le 8(\max\{\gamma_{13},1\}/\gamma_{13})\epsilon+8(\bar{\gamma}_{13}/\gamma_{13})\omega_m R_0^2, \qquad (5.46\ )
\end{aligned}
$$

where we used the fact that the constant C in the density $\tilde{e}_k(u)$ is sufficiently large, an inequality (2.7) and an estimation

$$1/\sqrt{\tilde{e}_0/2}<\rho_0\le r_1/2=\delta R/2<R_0.$$

Take $\epsilon > 0$ as small, dependently of $\gamma_1, \gamma_4, \gamma_{13}, p$ and m, and R_0 as small, dependently of $\gamma_1, \gamma_4, \gamma_{13}, \bar{\gamma}_{13}, p$ and m, so that we obtain the contradiction from (5.44) and (5.46) with (5.36). Therefore we conclude that $r_0 \leq 1$. By choice of σ_0, this implies

$$\max_{0 \leq \sigma \leq r_1} \{(r_1 - \sigma)^2 \sup_{Q_\sigma} \tilde{e}_k(u)\} \leq 4\rho^2 \tilde{e}_0 = 4r_0^2 \leq 4. \tag{5.47}$$

We choose $\sigma = (1/2)r_1 = (\delta/2)R$ in (5.47) and divide the both side of the resulting inequality by $(\delta R/2)^2$ to obtain (5.35).

6 Proof of Theorem.

From our energy inequality (3.6), we observe that, for an initial data $u_0 \in W^{1,2}(M, N)$, there exist a subsequence of $\{u_k\}$ and a map $u \in L^\infty([0, +\infty); W^{1,2}(M, R^n)) \cap W^{1,2}((0, +\infty); L^2(M, R^n))$ such that, taking the limit $k \to \infty$, then we have

$$Du_k \to Du \quad \text{weakly* in } L^\infty([0, \infty); L^2(M, R^n)) \ , \tag{6.1 }$$

$$\partial_t u_k \to \partial_t u \quad \text{weakly in } L^2((0, \infty) \times M, R^n)) \ , \tag{6.2 }$$

$$u_k \to u \quad \text{weakly in } L^2_{loc}((0, \infty); W^{1,2}(M, R^n)) \ . \tag{6.3 }$$

From (6.3), we also find that

$$u_k \to u \quad \text{almost everywhere in } (0, +\infty) \times M. \tag{6.4}$$

Again, by (3.6), we have

$$\mathrm{dist}(u_k, N) \to 0 \quad \text{in } L^2_{loc}((0, +\infty); L^2(M)). \tag{6.5}$$

From (6.4) with (6.5) and (3.6) with (6.1), (6.2) we obtain that

$$u \in N \quad \text{almost everywhere in } (0, +\infty) \times M, \tag{6.6 }$$

$$\sup_{0 \leq T < +\infty} \left(\int_{(0,T) \times M} |\partial_t u|^2 dM dt + \int_{\{T\} \times M} f(Du|^2) dM \right) \leq I(u_0), \tag{6.7 }$$

where we used the convexity of $f(\tau^2)$ on τ.

We now define singular set for the weak limit u which is obtained as above. Let ϵ_0, R_0 are be constants determined in Lemma 5.3. Then let

$$\Sigma = \{z_0 = (t_0, x_0) \in (0, +\infty) \times M : \liminf_{k \to \infty} \int_{(t_0 - (2R)^2, t_0 - R^2) \times B_{R_M}} e_k(u_k) G_{z_0} \phi^2 dM dt \geq \epsilon_0$$
$$\text{for any } R, 0 < R < R_0 \ \}. \tag{6.8 }$$

For $t_0 \in (0,+\infty)$ and R, $0 < R < R_0$, let

$$\Sigma_R^{t_0} = \{x_0 \in M : \liminf_{k\to\infty} \int_{(t_0-(2R)^2, t_0-R^2)\times\mathcal{B}_{R_M}} e_k(u_k) G_{z_0}\phi^2 dM dt \geq \epsilon_0\},$$
$$\Sigma^{t_0} = \cap_{0<R<R_0} \Sigma_R^{t_0}. \tag{6.9 }$$

Here note that $\Sigma = \cup_{t_0\in(-R_M^2,0)}\Sigma^{t_0}$. We are able to argue with our monotonicity formula (5.2) similarly as in [15] (also refer to [3]) to find that

$$\Sigma \text{ and } \Sigma^{t_0} \text{ are closed for any } t_0 \in (0,+\infty) \tag{6.10}$$

and to obtain an estimation on Hausdorff measure of a set Σ with respect to the parabolic metric δ.

$$\mathcal{H}^m_{\mathrm{loc}}(\Sigma) < +\infty, \quad \mathcal{H}^{m-2}(\Sigma^{t_0}) < +\infty \text{ for any } t_0 \in (0,+\infty). \tag{6.11}$$

Now we explain the outline of the proof to show that the limit u is weak solution to (1.7) and (1.8) (refer to [11] for details). We are able to argue similarly as in [[2], Page 93-95].

At first we show that u satisfies (1.7) almost everywhere in a local region Q around the point z_0 in the complement of the singular set Σ. Here the key lemma is Lemma 5.3. We obtain from our Bochner formula (5.2) and the assertion of Lemma 5.3 that

$$\left\{k\frac{d}{du}\chi(\mathrm{dist}^2(u_k, N))\right\} \text{ is bounded in } L^2_{loc}(Q). \tag{6.12}$$

Then, from (3.2), it follows that

$$\{\Delta_M^f u_k\} \text{ is bounded in } L^2_{loc}(Q). \tag{6.13}$$

(6.12) and (6.13) imply the existence of subsequence $\{u_k\}$ such that

$$k\frac{d}{du}\chi(\mathrm{dist}^2(u_k, N)), \quad \Delta_M^f u_k \quad \text{converge weakly in } L^2_{loc}(Q). \tag{6.14}$$

We find from our energy inequality (3.6) and (6.14) that there exists a function $A \in L^2_{loc}(Q, R^{mn})$ with $DA \in L^2_{loc}(Q)$ such that, almost everywhere in Q, u satisfies the equation replaced the principal term in (1.7) by $\mathrm{div}A$, where we make a geometrical observation with (6.14).

Finally, noting that the operator Δ_M^f is monotone in $L^2_{loc}((0,+\infty); W^{1,2}(M, R^n))$, we observe from Lemma 3.1 with (6.2) and (6.3) that u is a weak solution to (1.7) and (1.8) satisfying the energy inequality (1.10), where we use a usual covering lemma with the estimation (6.11) on Hausdorff measure of the singular set Σ.

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