

## Introducing the Hardline in Proof Theory

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G. Gentzen [G3] published his new version of consistency proof for first order number theory in 1938. He had already had two consistency proofs [G1] and [G2]. The first used *constructive but rather abstract notion of functionals*. In the second he had first introduced *transfinite ordinals* in proof theory. Although he formulated the result as a consistency proof, his interest seems to involve a taking off from Hilbert's program. As to this turning G. Kreisel [K] p. 262 wrote:

..., by introducing a *quantitative ordinal measure* he (=Gentzen) forces us to pay attention to *combinatorial complexity*<sup>1</sup> and thereby makes it at least more difficult for us to slip into an abstract reading.

It seems that the purpose of the third "Neue Fassung" is to make a lucid exposure of this combinatorial complexity which Gentzen discovered in *finite proof figures* of number theory.

G. Takeuti followed this idea and developed a proof theory of systems of second order arithmetic including  $\Pi_1^1$ -Comprehension Axiom,  $\Pi_1^1 - CA$ .

We follow in the wake of Gentzen and Takeuti. Proof theory à la Gentzen proceeds as follows;

(G1) Let  $P$  be a proof whose endsequent has a restricted form. Define a reduction procedure  $r$  which rewrites such a proof  $P$  to yield another proof  $r(P)$  provided that  $P$  has not yet reduced to a certain canonical form.

(G2) From the structure of the proof  $P$ , we abstract a structure related to this procedure  $r$  and throw irrelevant residue away. Thus we get a finite figure  $o(P)$ .

We call the figure  $o(P)$  the *ordinal diagram* (abbr. by o.d.'s) after G. Takeuti [T]. Let  $\mathcal{O}$  denote the set of o.d.'s.

(G3) Define a relation  $<$  on  $\mathcal{O}$  so that  $o(r(P)) < o(P)$ .

(G4) Show the relation  $<$  on  $\mathcal{O}$  is well founded.

Usually  $<$  is a linear ordering and hence  $(\mathcal{O}, <)$  is a notation system for ordinals.

This description is not acute. In fact (G1)-(G4) interact each other. For example (G1) is influenced by (G3) and this by (G4).

In this paper we expound some basic ideas of proof theory for theories of ordinals  $\sigma$  such that there are many  $\sigma$ -stable ordinals below  $\sigma$ . From this we get the proof theoretic ordinals of subsystems of second order arithmetic, e.g.,  $\Sigma_3^1 - DC + BI$ . The details will be reported in [A2], [A3], [A4].

In §2 we expound proof theory for  $\Pi_3$ -reflecting and  $\Pi_4$ -reflecting ordinals in some detail. In §3 theories for ordinals  $\sigma$  having many  $\sigma$ -stable ordinals below are analysed.

For more on the aims and another approach to proof theory of strong theories, see M. Rathjen [R1] and [R2].

### 1 $\Pi_2^\Omega$ -ordinal of a theory

G. Jäger [J] has shifted an object of proof-theoretic study to set theories from second order arithmetic.

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<sup>1</sup>This emphasis is made by the author.

**Definition 1** ( $\Pi_2^\Omega$ -ordinal of a theory) Let  $T$  be a recursive theory of sets such that  $KP\omega \subseteq T \subseteq ZF+V=L$ , where  $KP\omega$  denotes Kripke-Platek set theory with the Axiom of Infinity. For a sentence  $A$  let  $A^{L_\alpha}$  denote the result of replacing unbounded quantifiers  $Qx$  ( $Q \in \{\forall, \exists\}$ ) in  $A$  by  $Qx \in L_\alpha$ . Here for an ordinal  $\alpha \in Ord$   $L_\alpha$  denotes an initial segment of Gödel's constructible sets. Let  $\Omega$  denote the (individual constant corresponding to the) ordinal  $\omega_1^{CK}$ . If  $T \not\vdash \exists \omega_1^{CK}$ , e.g.,  $T = KP\omega$ , then  $A^{L_\Omega} =_{df} A$ . Define the  $\Pi_2^\Omega$ -ordinal  $|T|$  of  $T$  by

$$|T| =_{df} \inf\{\alpha \leq \omega_1^{CK} : \forall \Pi_2 \text{ sentence } A(T \vdash A^{L_\alpha} \Rightarrow L_\alpha \models A)\} < \omega_1^{CK}$$

Here note that  $|T| < \omega_1^{CK}$  since we have

$$\forall \Pi_2 \text{ sentence } A(T \vdash A^{L_\Omega} \Rightarrow L_\Omega \models A)$$

and  $\Omega = \omega_1^{CK}$  is recursively regular, i.e.,  $\Pi_2$ -reflecting.

G. Jäger [J] shows that  $|KP\omega|$  = Howard ordinal and G. Jäger and W. Pohlers [J-P] gives the ordinal  $|KPi|$ , where  $KPi$  denotes a set theory for recursively inaccessible universes. Also see Buchholz and Schütte [B-S] and Schütte [S] for related results. These include and imply proof-theoretic ordinals of subsystems of second order arithmetic corresponding to set theories.

We will develop proof theory for theories of ordinals: Let  $\mathcal{L}_0$  denote the first order language whose constants are:  $=$  (equal),  $<$  (less than),  $0$  (zero),  $1$  (one),  $+$  (plus),  $\cdot$  (times),  $j$  (pairing),  $()_0, ()_1$  (projections, i.e., inverses to  $j$ ). For each  $\Delta_0$  (=bounded) formula  $\mathcal{A}(X, a, b)$  (a binary predicate  $X$ ) we introduce a binary constant  $R^A$  such that

$$b \in R^A \Leftrightarrow_{df} R^A(a, b) \Leftrightarrow \mathcal{A}(R^A_{<a}, a, b)$$

with  $R^A_{<a} = \sum_{x < a} R^A_x = \{(x, y) : x < a \ \& \ y \in R^A_x\}$ .

Let  $F : Ord \rightarrow L$  denote (a variant of) the Gödel's enumeration of constructible sets. Then one can define relations  $\varepsilon$  and  $\equiv$  on  $Ord$  such that

$$\alpha \varepsilon \beta \Leftrightarrow F'\alpha \in F'\beta ; \alpha \equiv \beta \Leftrightarrow F'\alpha = F'\beta.$$

and these relations  $\varepsilon$  and  $\equiv$  are definable by  $\Delta_0$  formulae in the language  $L_0 \cup \{R^A\}$ .

Thus, in principle, one can define a theory  $T^{Ord}$  of ordinals for each set theory  $T$  by interpreting  $\in$  and  $=$  as  $\varepsilon$  and  $\equiv$ , resp. In place of  $T^{Ord}$  we consider a theory  $T_n$  of  $\Pi_n$ -reflecting ordinals.

**Definition 2** (Aczel and Richter [A-R]) Let  $X \subseteq Ord$  denote a class of ordinals and  $\Phi$  a set of formulae in the language of set theory (or the language of theories of ordinals). Put  $X|\alpha =_{df} \{\beta \in X : \beta < \alpha\}$ . We say that an ordinal  $\alpha \in Ord$  is  $\Phi$ -reflecting on  $X$  if

$$\forall A \in \Phi \text{ with parameters from } L_\alpha [L_\alpha \models A \Rightarrow \exists \beta \in X|\alpha (L_\beta \models A)]$$

If a parameter  $\gamma < \alpha$  occurs in  $A$ , then it should be understood that  $\gamma < \beta$ .

$\alpha$  is  $\Phi$ -reflecting if  $\alpha$  is  $\Phi$ -reflecting on the class of ordinals  $Ord$ .

## 2 $\Pi_3$ and $\Pi_4$ reflection

Let us explain how to design a notation system  $O(\Pi_3)$  of ordinals (its elements are called *ordinal diagrams* abbr. by o.d.'s) and show

$$\forall \Pi_2 A(T_3 \vdash A \Rightarrow \exists \alpha \in O(\pi_3) |\Omega (= \{\alpha \in O(\pi_3) : \alpha < \Omega\}) \text{ s.t. } A^\alpha).$$

■■■  $T_3$  is formulated in Tait's logic calculus, i.e., one-sided sequent calculus and  $\Gamma, \Delta \dots$  denote a *sequent*, i.e., a finite set of formulae.  $T_3$  has the inference rule ( $\Pi_3 - rfl$ ):

$$\frac{\Gamma, A \quad \neg A^b, \Gamma}{\Gamma} (\Pi_3 - rfl)$$

where  $A \equiv \forall x \exists y \forall z B$  with a bounded formula  $B$  and the eigenvariable  $b$ .

So ( $\Pi_3 - rfl$ ) says  $A \rightarrow \exists b A^b$ .<sup>2</sup>

To deal with the rule ( $\Pi_3 - rfl$ ) we introduce a new rule:

$$\frac{\Gamma, A}{\Gamma, A^{\alpha_0}} (cp)$$

<sup>2</sup>For simplicity we suppress the parameter. Correctly  $\forall u(A(u) \rightarrow \exists z(u < z \ \& \ A^z(u)))$ .

where  $A$  is a  $\Pi_3$ -sentence as above.

We need to compute an o.d.  $\alpha_0 < \pi$  in order to replace the  $(\Pi_3 - rfl)$  by a  $(cut)$ :

$$\frac{\frac{\Gamma, A}{\Gamma, A^{\alpha_0}} (cp) \quad [b := \alpha_0]}{\Gamma} \frac{\neg A^{\alpha_0}, \Gamma}{\Gamma} (cut)$$

Firstly we throw  $0$  and  $\pi$  into  $O(\Pi_3)$ . The o.d.  $\pi$  corresponds to the first  $\Pi_3$ -rfl ordinal. Let  $O(\Pi_3)$  be closed under  $+$  and the Veblen function  $\varphi$ . The Veblen function  $\varphi$  is needed for treating the constant  $R^A$ . Let  $\mathcal{R}$  denote the set of o.d.'s corresponding to recursively regular ordinals.

We have learnt the following fact from the proof theory for the universes with many recursively regular ordinals: In general, if  $\sigma$  is recursively regular, then we have to introduce a collapsing  $(\sigma, \alpha) \mapsto d_\sigma \alpha$ .

For example, it suffices to have two steps collapsings for recursively Mahlo ordinals:

$(\mu, \alpha) \mapsto d_\mu \alpha = \sigma$  and  $(\sigma, \beta) \mapsto d_\sigma \beta$  with the first recursively Mahlo ordinal  $\mu$ .

The relation  $\alpha < \beta$  is defined so as to hold:

(< 1)  $d_\sigma \alpha < \sigma$

(< 2)  $K_\sigma \alpha < d_\sigma \alpha$

(< 3)  $K_\sigma \alpha \leq \alpha$

(< 4)  $\alpha < \sigma \ \& \ K_\sigma \alpha < d_\sigma \beta \Rightarrow \alpha < d_\sigma \beta$

where  $K_\sigma \alpha$  denotes the finite set of subdiagrams  $\beta$  of  $\alpha$  such that, in the construction of  $\alpha$ ,  $\beta$  is a last collapse of  $\sigma$ , i.e.,

$$\exists \{\sigma_i\}_{i \leq n} \forall i < n (\sigma = \sigma_0 \ \& \ \sigma_{i+1} = d_{\sigma_i} \ \& \ \sigma_n = \beta)$$

The first candidate to  $\alpha_0$  is  $d_\pi \alpha$  with  $\alpha = o(\Gamma, A)$ , where  $o(\Gamma)$  denotes the o.d. assigned to the sequent  $\Gamma$ . But this does not work. Consider a proof with nested rules  $(\Pi_3 - rfl) J, J_1$ :

$$\frac{\frac{A_1 \quad \neg A_1^{b_1}}{J_1} \quad \vdots}{A} \frac{\neg A^b}{J}$$

$$A_1 \equiv \forall x_1 \exists y_1 \forall z_1 B_1, \quad A \equiv \forall x \exists y \forall z B.$$

First replace the lower  $(\Pi_3 - rfl) J$  by a  $(cp) K_0$  followed by a  $(cut) I$ :

$$\frac{\frac{A_1 \quad \neg A_1^{b_1}}{J_1} \quad \vdots}{\frac{A}{A^{\alpha_0}} K_0} \frac{[b := \alpha_0]}{\neg A^{\alpha_0}} I \quad \text{Fig.1}$$

with an o.d.  $\alpha_0 < \pi$ , e.g.,  $\alpha_0 = d_\pi \alpha$ .

Then do the same thing to the above  $(\Pi_3 - rfl) J_1$ :

$$\frac{A_1}{A_1^{\alpha_1}} (cp) \quad [b_1 := \alpha_1] \quad \frac{\neg A_1^{\alpha_1}}{\neg A_1^{\alpha_1}}$$

We are forced to have  $\alpha_1 < \alpha_0$  since  $\alpha_1$  may be substituted for  $y$  in  $\exists y \forall z B$ , i.e.,  $\exists y < \alpha_0 \forall z < \alpha_0 B$ . But the innermost unbdd universal quantifier  $\forall z$  in  $A$  causes troubles since any o.d.  $\beta < \alpha_0$  may be substituted for  $z$ , e.g.,  $\beta \geq \alpha_1$ , and this destroys the case

$$\frac{\frac{\forall z_1 B_1(\beta)}{\exists y_1 \forall z_1 B_1(y)} (\exists)}{\exists y_1 < \alpha_1 \forall z_1 < \alpha_1 B_1} (cp)$$

We cannot anticipate that what o.d.  $\beta$  is substituted for  $z$  except  $\beta < \alpha_0$  and  $\beta$  comes from the right upper part of the  $(cut) I$ .

How to get rid of this difficulty? Our answer is to *iterate collapsings*: Put  $\alpha_0 = d_\pi \alpha$  and a  $(cp), K_1$  resolving the  $(\Pi_3 - rfl) J_1$  situates below  $I$ :

$$\begin{array}{c}
 A_1 \\
 \vdots \\
 A, A_1 \\
 \hline
 A^{\alpha_0}, A_1^{\alpha_0} \quad K'_0 \quad \beta \\
 \vdots \\
 \neg A^{\alpha_0} \\
 \hline
 I \\
 \hline
 \frac{A_1^{\alpha_0}}{A_1^{\alpha_1}} \quad K_1 \quad [b_1 := \alpha_1] \\
 \hline
 \neg A_1^{\alpha_1} \quad (cut) \quad Fig.2
 \end{array}$$

with  $\alpha_1 = d_{\alpha_0}\beta_1$ ,  $\beta_1 = o(A_1^{\alpha_0})$

Then  $\beta < \alpha_0$  &  $K_{\alpha_0}\beta < \alpha_1 \Rightarrow \beta < \alpha_1$  is seen from ( $< 4$ ).  $K_{\alpha_0}\beta < \alpha_1$  is satisfied since  $K_{\alpha_0}\beta \subseteq K_{\alpha_0}\beta_1 < d_{\alpha_0}\beta = \alpha_1$  by ( $< 2$ ).<sup>3</sup>

In this way we reduce proof figures. The problem is that we have an infinite iteration of collapsings in  $O(\pi_3)$ :  $\pi \mapsto d_\pi \mapsto d_{d_\pi} \mapsto \dots$ . Thus we have readily an infinite decreasing sequence by the requirement ( $< 1$ )  $d_\sigma\alpha < \sigma$ . We have to kill this infinite sequence. Let us examine what changes when we pass from  $\alpha_0$  to  $\alpha_1$ .

Observe that the upper part of the (cp)  $K_0$  in Fig.1 becomes simpler in Fig.2, i.e., the (cp)  $K'_0$ . This reflects to o.d.'s so that  $o(A, A_1) = o(K'_0) < o(K_0) = o(A)$ . Therefore when we iterate collapsings, i.e., build a tower of rules (cp) growing downwards, the upper part of the topmost (cp) becomes simpler, i.e., o.d. decreases because of resolving ( $\Pi_3 - rfl$ ). Hence when we introduce an o.d.  $d_\sigma\alpha$  from  $(\sigma, \alpha)$  we attach the o.d.  $\mu$  corresponding to this upper part:  $(\sigma, \alpha, \mu) \mapsto d_\sigma^\mu\alpha$ . We call the o.d.  $\mu$  the  $q$ -part of the o.d.  $d_\sigma^\mu\alpha$  and denote  $\mu = st(d_\sigma^\mu\alpha)$ .<sup>4</sup> And require that:

$$st(d_\sigma^\mu\alpha) < st(\sigma) \text{ if } \sigma \neq \pi, \text{ i.e., } \mu < \nu \text{ for } d_{d_\nu^\mu\beta}^\mu\alpha \quad (1)$$

Then it may be the case that any infinite collapsing processes are killed by this proviso (1).

Nonetheless this is not the end of the story. First  $\mu = st(d_\sigma^\mu\alpha) \geq \pi$  in general and so a well ordering proof may be difficult. Further, on the side of proof figures, the proviso (1) means that we have to pinpoint, for each (cp), the unique succession of rules (cp), called the *chain*, which describes how to introduce the (cp): For each

$$\frac{A^\sigma}{A^{d_\sigma^\mu\alpha}} (cp)_{d_\sigma^\mu\alpha}^\sigma,$$

pinpoint the unique chain

$$\frac{A^\pi}{A^{\sigma_1}} (cp)_{\sigma_1}^\pi, \frac{A^{\sigma_1}}{A^{\sigma_2}} (cp)_{\sigma_2}^{\sigma_1}, \dots, \frac{A^\sigma}{A^{d_\sigma^\mu\alpha}} (cp)^\sigma$$

such that  $\sigma_1 = d_\pi^{\mu_1}\alpha_1, \sigma_2 = d_{\sigma_1}^{\mu_2}\alpha_2, \dots$

These (cp)'s are connected or related each other by collapsing. And furthermore it must be the case  $o(\text{the upper part of the topmost } (cp)^\pi) \leq \mu$ , and this topmost (cp) <sup>$\pi$</sup>  must be determined uniquely from each (cp) <sup>$\sigma$</sup> . For otherwise suppose there are two chains for a (cp):

$$\begin{array}{ccc}
 (\Pi_3 - rfl) I_0 & & (\Pi_3 - rfl) I_1 \\
 \frac{A^\pi}{A^{\sigma_1}} \text{ top } J_0 & & \frac{A^\pi}{A^{\sigma_1}} \text{ top } J_1 \\
 & \vdots & \text{chains} \\
 & & \frac{A^\sigma}{A^{d_\sigma^\mu\alpha}} (cp)_{d_\sigma^\mu\alpha}^\sigma
 \end{array}$$

$J_0, J_1$  are topmost ones of chains. Even if we have  $o(J_0), o(J_1) \leq \mu$ , there may be ( $\Pi_3 - rfl$ )'s  $I_0$  and  $I_1$  above  $J_0$  and  $J_1$ , resp. Here we cannot anticipate which one of  $o(J_0)$  and  $o(J_1)$  is bigger. So the proviso (1) breaks down.

To retain the uniqueness of the chain, i.e., not to branch or split a chain, we have to be careful in resolving rules with two uppersequents.

Let us examine more closely the situation since this is instructive for  $\Pi_4 - rfl$ . Our guiding principles are:

**(ch1)** For any  $\frac{A^\sigma}{A^\tau} (cp)_\tau^\sigma$  with  $\tau = d_\sigma^\mu\alpha$ , if an o.d.  $\beta$  is substituted for an existential quantifier  $\exists y < \sigma$  in  $A^\sigma$ , i.e.,  $\beta$  is a realization for  $\exists y < \sigma$ , then  $\beta < \tau$ , and

**(ch2)** Resolving rules such as (cut) must not branch a chain.

<sup>3</sup>  $d_\pi\alpha \in \mathcal{R}$  since, in general, the closure ordinal  $\beta$  is recursively regular with  $A^\pi \Rightarrow \exists \beta < \pi A^\beta$  for a  $\Pi_3 A$ , cf. [A-R].

<sup>4</sup> *st* stands for *Stufe* or *stage*.

1) First resolve a  $(\Pi_3 - rfl)$ :

$$\frac{\frac{A}{A^\sigma} (cp)_\sigma^\pi}{\neg A^\sigma} J_0 (cut) \quad Fig.3$$

with  $A \equiv \forall x_1 \exists x_2 \forall x_3 A_3$ ,  $\sigma = d_\pi^\mu \alpha$ .

Then resolve the  $(cut) J_0$ :

$$\frac{\frac{A}{A^\sigma} \quad \frac{\neg A^\sigma, \neg A_1^\sigma}{\neg A_1^\sigma} \quad \frac{A_1}{A_1^\sigma} I_0}{\frac{\neg A_1^\sigma}{A_1^\sigma} J_1 (cut)} \quad Fig.4$$

with a  $\Sigma_2 A_1$ .

2) Second resolve a  $(\Pi_3 - rfl)$  above the  $(cp) I_0$  and a  $(cut)$  as in 1):

$$\frac{\frac{\frac{P_5}{A_1, B} \quad \frac{\neg A_1^\sigma}{A_1^\sigma, B^\sigma} \tilde{I}_0 (cp)_\sigma^\pi}{\frac{B^\sigma}{B^\tau}} \quad \frac{\frac{A_1, \neg B^\tau, \neg B_1^\tau}{A_1^\sigma, \neg B^\tau, \neg B_1^\tau} \quad \frac{A_1, B_1}{A_1^\sigma, B_1^\sigma} (cp)_\sigma^\pi}{\frac{\neg B_1^\tau}{B_1^\sigma} (cp)_\tau^\sigma} J_1}{\frac{\neg B_1^\tau}{B_1^\sigma} (cp)_\tau^\sigma} \quad Fig.5$$

with  $\tau = d_\sigma^\nu \beta$ , a  $\Sigma_2 B_1 \equiv \exists y_2 \forall y_3 B_3$ .

After that resolve the  $(cut) J_1$ :

$$\frac{\frac{\frac{P_6}{A_1, B} \quad \frac{\neg A_1^\sigma}{A_1^\sigma, B^\sigma} \tilde{I}_0}{\frac{B^\sigma}{B^\tau}} \quad \frac{\frac{A_1, B_1, A_2}{A_1^\sigma, B_1^\sigma, A_2^\sigma} \quad \frac{A}{A^\sigma} \quad \frac{\neg A^\sigma, \neg A_2^\sigma}{\neg A_2^\sigma} J'_0}{\frac{\neg B_1^\tau}{B_1^\sigma} \quad \frac{B_1^\sigma}{B_1^\tau}} \quad Fig.6$$

Then resolve the  $(cut) J'_0$ :

$$\frac{\frac{\frac{P_7}{A_1, B} \quad \frac{\neg A_1^\sigma}{A_1^\sigma, B^\sigma} \tilde{I}_0}{\frac{B^\sigma}{B^\tau}} \quad \frac{\frac{B_1^\sigma, A_2^\sigma}{\neg B_1^\tau, \neg B_1^\tau} \quad \frac{\frac{(\Pi_3 - rfl) H}{\dots} \quad \frac{\tilde{A}_1}{\tilde{A}_1} (cp)_\sigma^\pi I'_0}{\neg A_2^\sigma, \neg \tilde{A}_1^\sigma} J_2 (cut)}{\frac{B_1^\sigma}{B_1^\tau} I_1} K \quad Fig.7$$

3) Thirdly assume that we resolve a  $(\Pi_3 - rfl) H$  above the  $(cp)_\sigma^\pi I'_0$ . We introduce a new  $(cp)_\rho^\sigma I'_1$  with  $\rho = d_\sigma^\eta \gamma$  immediately above the  $(cut) J_2$ . Then the new  $(cp)_\rho^\sigma I'_1$  is introduced after the  $(cp)_\tau^\sigma I_1$  and so  $\rho = d_\sigma^\eta \gamma < \tau$ . Hence a new  $(cut) K'$  is introduced below the  $(cut) K$ :



needed for a well ordering proof.

Let us try to prove that there is no infinite succession  $\pi = \sigma_0, \sigma_1, \dots$  of collapsing with  $\sigma_{n+1} = d_{\sigma_n}$ . Assume such an infinite sequence exists. It suffices to show, then, there would exist an infinite subsequence  $\{\sigma_{n_i}\}_{i \in \omega}$  such that

$$\forall i \in \omega [st_4(\sigma_{n_{i+1}}) < st_4(\sigma_{n_i})]$$

Such a subsequence  $\{\sigma_{n_i}\}$  amounts to a subseries  $\{I_{n_i}\}$  of the infinite chain  $\{I_n\}$  such that each  $I_{n_i}$  is introduced as a resolvent of a  $(\Pi_4 - rfl)$  above  $I_{n_0}$ .

Consider the case when

$$\exists \tau [\#\{n \in \omega : rg_3(\sigma_n) = \tau\} = \aleph_0], \text{ i.e., } \exists \{\sigma_{n_i}\} \forall i \in \omega [rg_3(\sigma_{n_i}) = \tau]$$

Then by the proviso (3) we would have

$$\forall i \in \omega [st_3(\sigma_{n_{i+1}}) < st_3(\sigma_{n_i})]$$

We can expect this is not the case. And what else? There may be the case

$$\forall \tau [\#\{n \in \omega : rg_3(\sigma_n) = \tau\} < \aleph_0]$$

This means that the new merging points go downwards unlimitedly. For example in *Fig.9* a new succession with a merging point  $(cp)_\rho^\tau I_2$  arises by resolving a *(cut)* below the  $(cp)_\tau^\sigma I_1$ , i.e.,  $\tilde{I}_0 - I_1' - I_2 - I_3 (cp)_\kappa^\rho$  with a  $\kappa = d_\rho^{\lambda \pi \xi \tau} \delta$ . But in this case we have

$$\lambda = st_4(\kappa) < st_4(\tau) = \nu$$

$st_4(\kappa)$  corresponds to the upper part  $P_5$  of a  $(cp)_\sigma^\tau \tilde{I}_0$  in *Fig.5*, when the  $(cp)_\tau^\sigma$  was originally introduced. This part  $P_5$  is unchanged up to *Fig.9*:

$P_5 = P_6 = P_7 = P_9$ . Roughly speaking,  $\tilde{I}_0 - I_1' - I_3$  can be regarded as a  $\Pi_3$ -series  $I_0 - I_1 - I_3$ . In this way even if the new merging points grow downwards unlimitedly, we can find a subsequence  $\{\sigma_{n_i}\}$  such that  $st_4(\sigma_{n_{i+1}}) < st_4(\sigma_{n_i})$ . Thus any succession of collapsings terminates in a finite number of steps.

Once  $\Pi_4 - rfl$  can be analyzed, it is not so hard to treat  $\Pi_n - rfl$  ( $n < \omega$ ) and further  $\Pi_\alpha - rfl$  for a given transfinite  $\alpha < \text{the least } \Pi_\alpha - rfl \text{ ordinal}$ .

Now is the time for turning to stability from reflection.

### 3 Ordinals $\sigma$ having $\sigma$ stable ordinals below

**Definition 3** For  $\alpha, \beta \in Ord$  with  $\alpha < \beta$ ,  $\alpha$  is  $\beta$ -stable if  $L_\alpha \prec_{\Sigma_1} L_\beta \Leftrightarrow_{df} \forall \Sigma_1 A$  in  $L_\alpha (L_\beta \models A \Leftrightarrow L_\alpha \models A)$

The reason for this turning to stability is that  $\Sigma_2^1$ -Comprehension Axiom is interpretable in a universe  $L_\beta$  such that  $L_\beta$  has  $\beta$ -stable ordinals.

We consider a baby case, i.e., ordinals  $\sigma^+$  such that  $\sigma$  is  $\sigma^+$ -stable. Here recursion theoretic facts are helpful.

**Facts.** (cf. [A-R] and [M].) For a countabl  $\sigma$ ,

1.  $\sigma$  is  $\Pi_1^1$ -reflecting  $\Leftrightarrow \sigma$  is  $\sigma^+$ -stable.
2.  $\Pi_1^1$  on  $L_\sigma$  = inductive on  $L_\sigma = \Sigma_1$  on  $L_{\sigma^+}$ .

Let  $S_1^1$  denote a theory of ordinals  $\sigma^+$  and  $T_1^1$  a theory of ind-reflecting ordinals.

**Definition 4**  $S_1^1$  and  $T_1^1$

1. The language of  $S_1^1$  is  $\mathcal{L}_0 \cup \{R^A\} \cup \{\Omega, \pi\}$ . The axioms of  $S_1^1$  say that the universe  $\pi^+$  of  $S_1^1$  is  $\Pi_2$ -reflecting and the ordinal  $\pi$  is  $\pi^+$ -stable: for each  $\Sigma_1 A \forall u < \pi (A(u) \rightarrow A^\pi(u))$  or equivalently  $\forall u < \pi (A(u) \rightarrow \exists y < \pi (y > u \& A^y(u)))$ .

The corresponding rule runs as follows:

$$\frac{\Gamma, \neg(t < b < \pi \wedge A^b(t)) \quad t < \pi \wedge A(t), \Gamma}{\Gamma} \text{ (stbl)}$$

2. The language of  $T_1^1$  is the language of  $S_1^1$  plus  $\{I_{<}\}$ , where  $I_{<}$  is a ternary predicate constant: Fix an  $X$ -positive formula  $A \equiv A(X^+, a)$  in  $\mathcal{L}_0 \cup \{R^A\} \cup \{X\}$ . Let  $Mp$  denote the set of multiplicative principal numbers  $a \leq \pi$  and  $a^+$  the next admissible to  $a$ . Then the intended meaning of the constant  $I_{<}$  is given by:

$$\forall a \in Mp \forall b < a^+ [I_{<}_b^a = \bigcup_{d < b} I_d^a = \bigcup_{d < b} \{c < a : A^a(I_{<}_d^a, c)\}]$$

That is to say, for each  $a \in Mp$ ,  $a \leq \pi$  and  $b < a^+$ ,  $I_{<}_b^a$  is the inductively generated subset of  $a = \{c : c < a\}$  by the positive formula  $A$  on the model  $\langle a; +, \cdot, \dots, R^A, \dots \rangle$ , uniformly with respect to the multiplicative principal number  $a$ .

The axioms of  $T_1^1$  say that the universe  $\pi^+$  is  $\Pi_2$ -reflecting and the axiom  $(\Pi_1^1 - rfl)$ :

$$\forall c < \pi [c \in I_{<}_{\pi^+}^\pi \rightarrow \exists \beta \in (c, \pi) \cap Mp (c \in I_{<}_{\beta^+}^\beta)].$$

where  $c \in I_{<}_{a^+}^a \Leftrightarrow_{df} \exists z < a^+ A^a(I_{<}_z^a, c)$ .

Then it is not hard to see that  $S_1^1$  is interpretable in  $T_1^1$ : we can extract an interpretation from Chapter 9 in Moschovakis [M].

Before developing a proof theory for the theory  $S_1^1$ , we stay the theory  $T_1^1$  for a while since the latter is still a theory of reflecting ordinals and an analysis for it may be attainable from  $\Pi_\alpha$ -reflecting. We have intuitively:

$$\text{Predicative Analysis} : ID_1 = \Pi_\alpha\text{-reflecting} : T_1^1$$

and since the step from Predicative Analysis to  $ID_1$  requires a new dimension, an analysis for  $T_1^1$  would require a new twist too.

$$\frac{\neg(\alpha < b < \pi), \forall x < b^+ \neg A^b(I_{<}_x^b, \alpha) \quad \frac{A^\pi(I_{<}_\xi^\pi, \alpha)}{\exists x < \pi^+ A^\pi(I_{<}_x^\pi, \alpha)} (\exists)}{\neg A^\pi(I_{<}_\xi^\pi, \alpha)} (\Pi_1^1 - rfl) J$$

with  $\alpha \in I_{<}_{\pi^+}^\pi \equiv \exists x < \pi^+ A^\pi(I_{<}_x^\pi, \alpha)$ , etc.

First consider the easy case:

**Case1.**  $\xi < \pi$ : Then the above  $(\Pi_1^1 - rfl) J$  says that  $\pi$  is  $\Pi_\xi$ -reflecting. So define  $\sigma = d_\pi$  such that  $\xi, \alpha < \sigma < \pi$  and substitute  $\sigma$  for the variable  $b$ .

Second the general case:

**Case2.**  $\xi \geq \pi$ : Pick a  $\sigma = d_\pi$  as above and substitute  $\sigma$  for  $b$ . We need to compute a  $\xi'$  such that  $\sigma \leq \xi' < \sigma^+$  and resolve the  $(\Pi_1^1 - rfl) J$ :

$$\frac{\frac{\alpha \notin I_{<}_{b^+}^b \quad \alpha \in I_{<}_{\pi^+}^\pi, A^\pi(I_{<}_\xi^\pi, \alpha)}{A^\pi(I_{<}_\xi^\pi, \alpha)} J}{\frac{\neg A^\sigma(I_{<}_{\xi'}^\sigma, \alpha) \quad \frac{A^\pi(I_{<}_\xi^\pi, \alpha)}{A^\sigma(I_{<}_{\xi'}^\sigma, \alpha)} (cp)_\sigma^\pi I}{A^\sigma(I_{<}_{\xi'}^\sigma, \alpha)} (cut)}}$$

The problem is that we have to be consistent with the part



$$\frac{\neg A^\sigma(I_{<\xi'}^\sigma, \alpha)}{A^\pi(I_{<\xi}^\pi, \alpha)} \quad \frac{A^\pi(I_{<\xi}^\pi, \alpha)}{A^\sigma(I_{<\xi'}^\sigma, \alpha)}$$

This requires a function  $F : \xi \mapsto \xi'$  such that

(F1)  $F$  is order preserving, and in view of **Case1**,

(F2)  $F$  is identity on  $< \pi$ , i.e.,  $\xi \in \text{dom}(F) \mid \pi \Rightarrow F(\xi) = \xi$

(F3)  $\text{rng}(F) < \sigma^+$ .

Note that, here,  $\text{dom}(F)$  is a *proper subset* of  $\{\xi \in O(\pi_1^+) : \xi < \pi^+\}$  with a system  $O(\Pi_1^+)$  of o.d.'s for the theory  $T_1^1$ . We can safely set

$$\text{dom}(F) = \{\xi \in O(\pi_1^+) : K_\pi \xi < \sigma\}$$

, i.e., subdiagram  $\beta < \pi$  in  $\xi \in \text{dom}(F)$  is  $< \sigma$  since  $\text{dom}(F)$  is the set of o.d.'s that may occur in the upperpart of the  $(cp)_\sigma^\pi I$ . Especially we have

$$\text{dom}(F) \mid \pi = O(\pi_1^+) \mid \sigma$$

Can we take the function  $F$  as a collapsing function, e.g.,  $d_\pi$ ? The answer is no. We cannot expect for  $\xi, \zeta \in \text{dom}(F)$ , that  $\xi < \zeta \Rightarrow \xi \ll_\pi \zeta$  or something like an essentially less than relation. And what is worse is that the function  $F$  have to preserve atomic sentences in  $\mathcal{L}_0$ .

(F4)  $F$  preserves atomic sentences in  $\mathcal{L}_0$ , i.e., diagrams of  $\mathcal{L}_0$  models  $< \text{dom}(F); +, \cdot, \dots >$  and  $< \text{rng}(F); +, \cdot, \dots >$ .

To sum up (F1) – (F4),

(\*)  $F$  is an embedding from  $\mathcal{L}_0$  models  $< \text{dom}(F); +, \cdot, \dots >$  to  $< \text{rng}(F); +, \cdot, \dots >$  over  $O(\Pi_1^+) \mid \sigma$ .

Now our solution for  $F$  is a trite one: a *substitution*  $[\pi := \sigma]$ .

(F5)  $F(\xi) = \xi$  if  $\xi < \pi (\Leftrightarrow \xi < \sigma)$

(F6)  $F$  commutes with  $+$  and the Veblen function  $\varphi$ , e.g.,  $F(\xi + \zeta) = F(\xi) + F(\zeta)$ .

(F7)  $F(\pi) = \sigma$  and  $F(\pi^+) = \sigma^+$ .

(F8)  $F(d_{\pi^+}\beta) = d_{\sigma^+}F(\beta)$ .

Assume  $\pi < \xi < \pi^+$  with a strongly critical  $\xi$ . Such a  $\xi$  is of the form  $d_{\pi^+}\beta$  and is introduced when a  $(\Pi_2 - rfl)$  for the universe  $\pi^+$  is resolved. Then this  $F$  meets (\*), i.e., (F1): Note that we have

(F9)  $F(K_{\pi^+}\beta) = K_{\sigma^+}F(\beta)$ ,

and by definition  $d_{\pi^+}\beta < d_{\pi^+}\gamma \Leftrightarrow 1. \beta < \gamma \ \& \ K_{\pi^+}\beta < d_{\pi^+}\gamma$  or  $2. d_{\pi^+}\beta \leq K_{\pi^+}\gamma$  and similarly for  $\sigma^+$ .

In this way we can resolve a  $(\Pi_1^+ - rfl)$  by setting  $\xi' = F(\xi)$ : each o.d.  $\xi$  in the uppersequent of a  $(cp)$  is replaced by  $F(\xi)$  in the lowersequent.

Next consider the theory  $S_1^1$ .

$$\frac{\neg(\alpha < b < \pi \wedge A^b(\alpha)) \quad \frac{B(\xi, \alpha)}{\alpha < \pi \wedge A^{\pi^+}(\alpha)} (\exists)}{(\text{stbl}) J}$$

with  $A^{\pi^+}(\alpha) \equiv \exists x < \pi^+ B(x, \alpha)$ .

As in  $T_1^1$ , pick a  $\sigma = d_\pi$  and the substitution  $F = [\pi := \sigma]$ . Substitute  $\sigma^+$  for  $b$  and  $\xi' = F(\xi)$  for  $\xi$ .

$$\frac{\neg B(\xi', \alpha) \quad \frac{A^{\pi^+}(\alpha), B(\xi, \alpha)}{\neg A^{\sigma^+}(\alpha) \quad A^{\sigma^+}(\alpha), B(\xi', \alpha)} (cp)_\sigma^\pi I}{B(\xi', \alpha)}$$

When a universal quantifier  $\forall y$  occurs in  $B$ , then it must be a bounded one, say,  $\forall y < \xi' + \alpha$  since  $B$  is a bounded formula. An instance  $< \xi' + \alpha$  for the dual existential quantifier  $\exists y < \xi' + \alpha$  may come from the upperpart of  $\neg B(\xi', \alpha)$ . Then an inspection shows that the instance  $\in \text{rng}(F)$ , i.e., is of the form  $\zeta' = F(\zeta)$  for some  $\zeta \in \text{dom}(F)$ . Hence we substitute  $\zeta$  for the variable  $y$  in the upperpart of the  $(cp)_\sigma^\pi I$ .

In this way we can proceed and resolve consistently by (\*).

Next we consider an ordinal  $\pi$  which has many  $\pi$ -stable ordinals below. For example let  $\pi\omega$  be an  $\omega$  limit of  $\pi\omega$ -stable ordinals:

$$\pi\omega = \sup\{\pi n : n < \omega\} \ \& \ \forall n < \omega (\pi n \text{ is } \pi\omega\text{-stable})$$

The corresponding rule runs as follows:

$$\frac{\Gamma, \neg(\alpha < b < \pi n \wedge A^b(\alpha)) \quad \alpha < \pi n \wedge A^{\pi\omega}(\alpha), \Gamma}{\Gamma} \text{ (stbl)}_n$$

Assume  $\alpha < \pi n \wedge A^{\pi\omega}(\alpha)$  is a conclusion of an  $(\exists)$  with an auxiliary formula  $B(\xi, \alpha)$  with  $A^{\pi\omega}(\alpha) \equiv \exists x < \pi\omega B(x, \alpha)$ . As above we substitute  $\xi' = F(\xi)$  for  $\xi$  with  $F = [\pi n := \sigma]$  for a  $\sigma = d_{\pi n}\beta < \pi n$  with  $\beta = o(\alpha < \pi n \wedge A^{\pi\omega}(\alpha))$ .

This  $F$  have to mirror the situation of o.d.'s above  $\pi n$ , at least occurring above the right uppersequent  $\alpha < \pi n \wedge A^{\pi\omega}(\alpha)$ . Therefore we introduce (or better postulate the existence of ordinals corresponding to) o.d.'s  $\sigma m = F(\pi m) < \pi n$ ,  $\sigma < \sigma m$  for  $\omega \geq m > n$ . This o.d.  $\sigma m$  is a substitute for  $\pi m$  and so have to act as if it were  $\pi m$ . Further when we resolve a rule  $(stbl)_m$  with  $m > n$ , we introduce a  $\tau = d_{\pi m}\gamma$  with  $\pi n < \tau < \pi m$  and  $\tau k < \pi m$  for  $\omega \geq k > m$ . Thus we also have to introduce  $\tau' = d_{\sigma m}\gamma' = F(\tau) < \sigma m$  and  $\tau' k = F(\tau k) < \sigma m$ . Then  $\sigma < \tau' < \tau' k < \sigma m$ . Let  $O(2; \omega)$  denote the system of o.d.'s constructed in this way.

Here the consistency of the reduction procedure is not so problematic: these newly introduced o.d.'s are mirror images by the mirror  $F$ . Although  $\sigma m$  have to act as if it were  $\pi m$ , there need not be introduced a rule which says that  $\sigma m$  is  $\sigma\omega$ -stable. Hence as in  $\Sigma_1^1$  each instance term for an existential quantifier in  $\neg B(\xi', \alpha)$  is in  $rng(F)$ .

Rather the well foundedness of  $O(2; \omega)$  is problematic: consider a series  $\{\rho'_i\}$  such that  $\rho'_0 = \sigma m$  with  $n < m < \omega$ , and for each  $i > 0$ ,  $\rho'_i = \tau'_i(m+i)$  with  $\tau'_i = d_{\rho'_{i-1}}\beta_i$  for some  $\beta_i$ . Then we would have a ascending sequence followed by a descending sequence:

$$\sigma < \tau'_1 < \dots < \tau'_k < \tau'_{k+1} < \dots < \rho'_{k+1} < \rho'_k < \dots < \rho'_1 < \rho'_0 = \sigma m < \pi n$$

These o.d.'s came from the right upper part  $\alpha < \pi n \wedge A^{\pi\omega}(\alpha)$  of the rule  $(stbl)_n$  as mirror images by  $F$ . First of all preimages  $\{\rho_i\}$  of these were introduced and then these are introduced as  $\rho'_i = F(\rho_i)$ . These preimages were created to resolve the rule  $(stbl)_m$  and hence they were situated above the rule  $(stbl)_n$ . This means that  $\rho_i < \beta = o(\alpha < \pi n \wedge A^{\pi\omega}(\alpha))$  and, in fact a stronger  $\rho_i \ll_{\pi n} \beta$  holds. Therefore if we are in a situation that the o.d.  $\beta$  is secured, i.e., is in a well founded part of a subrelation of  $<$ , then so were the descending sequence  $\rho_i$ . This contradicts the well foundedness.

In this way we can prove that o.d.'s are well founded.

The whole argument works for the general case when we replace the order type  $\omega$  of stable ordinals by any ordinal. Thus we get a system of o.d.'s which represent a combinatorial complexity of proof figures in a theory for ordinals  $\sigma$  having many  $\sigma$ -stables. From this we also get an upper bound for the proof theoretic ordinal of a second order arithmetic for an iterated  $\Sigma_2^1 - CA$ .

1.  $\Sigma_2^1 - CA_0$ : The corresponding ordinal is a limit of ordinals  $\pi n$ ,  $n < \omega$  such that each  $\pi n$  is a limit of recursively regular ordinals and has  $n$   $\pi n$ -stable ordinals below. A system  $O(2; < \omega)$  of o.d.'s suffices.
2.  $\Sigma_2^1 - CA + BI$ : The ordinal  $\pi\omega$  is a limit of  $\pi\omega$ -stable ordinals, i.e., *nonprojectible ordinal*. The set theory  $KP\omega + \Sigma_1$  Separation is equivalent to this.  $O(2; \omega)$  suffices.
3.  $\Sigma_3^1 - DC_0$ : The ordinal is a limit of ordinals  $\pi a$ ,  $a < \omega^\omega$  such that each  $\pi a$  has a  $\pi a$ -stable ordinals below.  $O(2; < \omega^\omega)$  suffices.
4.  $\Sigma_3^1 - DC$ : The ordinal is a limit of ordinals  $\pi a$ ,  $a < \varepsilon_0$  such that each  $\pi a$  has a  $\pi a$ -stable ordinals below.  $O(2; < \varepsilon_0)$  suffices.
5.  $\Sigma_3^1 - DC + BI$ : This is included in the set theory  $KP\omega + \Pi_1$  Collection +  $V = L$ .

Let  $S_I$  denote a theory of ordinals  $I$  such that  $I$  is  $\Pi_2(St^+)$ -reflecting, where  $St$  denotes the set of stable ordinals below  $I$  and  $\Pi_2(St^+)$  the set of  $\Pi_2$  formulae  $A$  in the language  $\mathcal{L}_0 \cup \{St\}$  so that the predicate constant  $St$  occurs only positively in the formula  $A$ . Then the set theory  $KP\omega + \Pi_1 Collection + V = L$  is interpretable in  $S_I$ . A system  $O(2; I)$  is designed for  $S_I$ . In  $O(2; I)$  a constructor  $(I, \alpha) \mapsto d_I \alpha \in St$  generates  $I$ -stable ordinals.

Each of these systems of o.d.'s is shown to be best possible. For example we have

$$|\Sigma_2^1 - CA + BI| = |KP\omega + \Sigma_1 Separation| = O(2; \omega) | \Omega, \text{ etc.}$$

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