A Hierarchy of the Fragments of the System of
Inductive Definition (Preliminary Report)

Masahiro Hamano* and Mitsuhiro Okada†
Department of Philosophy
Keio University, Tokyo

1 introduction

Gentzen [7] proved the consistency of PA (Peano Arithmetic) by using the transfinite induction up to the first epsilon number $\epsilon_0$. Here $\epsilon_0$ is $\lim_k \omega_k$, where $\omega_0 = 0$ and $\omega_{k+1} = \omega^\omega$. Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than $\epsilon_0$, e.g., $\omega_k$ for any natural number $k$, is provable in $PA$.

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher $\omega$-tower $\omega_k$ is proved from the accessibility of $\omega_k$. Hence by considering Gentzen's work [7, 8] a natural question arises; does the hierarchy of $\omega$-towers, $\{\omega_k\}_{k=1,2,...}$, correspond exactly to a certain hierarchy of fragments of $PA$?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of $PA$, where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of $\xi$-iterated Inductive Definition $ID_{\xi}$ [6]. We first analyze in Section 2 Arai's optimal accessibility proof for $ID_{\xi}$ ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic $ID_{\xi}$, where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of $ID_{\xi}$ ([3]). In fact, for the upper bounds proof we use the fragments of classical $ID_{\xi}$ in terms of the nestedness complexity of classical negations. Since the fragments of $ID_{\xi}$ obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier $\exists$ by means of $\neg\forall$), our result for $ID_{\xi}$ corresponds to Mints' ([10]) for $PA$.

*Hamano@abelard.flet.mit.a.keio.ac.jp
†Okada@abelard.flet.mit.a.keio.ac.jp
2 Provability of transfinite inductions on $\omega(\xi, k, 0)$ in subsystems of $S_k(ID^i_\xi(U_0))$

Let $(I, \prec)$ be the well ordered system whose order type is ordinal $\xi + 1$. Arai [1] proved the well ordering of Takeuti's system of ordinal diagram $O(\xi + 1, 1)$ in the system $ID^i_\xi$ (the intuitionistic system of $\xi$-times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments $S_k(ID^i_\xi(U_0))$ based on the nestedness complexity of implications, and observe Arai's well ordering proof of [1] on these fragments.

Now we recall the definitions of $ID^i_\xi(U)$ and $ID^i_\xi$ of Feferman [6].

Definition 1 (System $ID^i_\xi(U)$ and $ID^i_\xi$, cf. Feferman [6])

For any positive operator form $U$, $ID^i_\xi(U)$ is obtained from $PA$ by adding the following axiom schemata.

\[(P\xi, 1) \quad \forall x < \xi(A(P^U_x, P^U_{\prec x}, x) \subseteq P^U_x)\]

\[(P\xi, 2) \quad \forall x < \xi(A(V, P^U_{\prec x}, x) \subseteq V \supset P^U_{\prec x} \subseteq V)\]

\[(TI) \xi \quad \text{Prog}[I, \prec, \forall] - (I \subseteq \forall)\]

where $P^U_{\prec x} := \{x, y)(x < a \land P^U_{\prec x} y\}$

$ID^i := \bigcup\{ID^i_\xi(U) \mid U \text{ is a positive operator form}\}$

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to $<_i$ for $i < \xi$ (cf. §26 [14]) by using the set constants $A_i$ which is definable in $ID^i_\xi(U_0)$ with the following $U_0$;

\[(A.1)_\xi \quad \forall i < \xi(\text{Prog}[F_i, <_i, A_i] \rightarrow A_i \subseteq V) \quad \text{for each abstract } V \text{ in } ID^i_\xi(U_0)\]

where $U_0$ is a $X$-positive operator form defined as $U_0(X, Y, i, \mu) := \mathcal{F}(i, \mu, Y) \land \forall \nu <_i \mu(\mathcal{F}(i, \nu, Y) \rightarrow X(\nu))$ where $\mathcal{F}(i, \mu, Y) := \forall k < i \nu \in C_k \mu Y(k, \nu), \text{Prog}[\alpha, \gamma, \beta] := \forall x(\alpha(x) \land \forall y(\gamma(y, x) \land \alpha(y) \rightarrow \beta(y)) \rightarrow \beta(x))$, and $F_i(\mu) := \forall j < i \nu \in C_j \mu A_j(\nu)$ (the intended meaning of $F_i(\mu)$ is that $\mu$ is an $i$-fan (cf. Definition 26.16 [14])).

Remember that $ID^i_\xi(U)$ has the mathematical induction of the following form;

\[(VJ) \quad V(0), \forall x(V(x) \rightarrow V(x')) \rightarrow V(t)\]

The above $ID^i_\xi(U_0)$ is the specific subsystem of the system $ID^i_\xi$ of Inductive Definition in which the induction schemata are used only for the accessibility predicate $A_i$ of ordinals.

We consider the subsystem $S_k(ID^i_\xi(U_0))$ of $ID^i_\xi(U_0)$ where each abstract $V$ in $A(2)_\xi$, $(TI)_\xi$ and $(VJ)$ is restricted to that of level $lv(V) \leq k$; where $lv(V)$ is defined by the definition below.

We introduce the notion of level of $A$ ($lv(A)$) for a formula $A$ to express, roughly speaking, the implicational complexity of $A$. We assume that the language contains only $\forall, \exists$ and $\land$ for the logical connectives in this section.

We first recall the degree $d$ of a formula in the language of $ID^i_\xi(U)$ defined in Arai [3], which intends to indicate how many times inductive definition is applied.

Definition 2 (cf. Def 2.4 in Arai [3])

- $d(t = s) = 0$ for all term $t, s$ and predicate variable $X$.

- $d(t^U_{\prec x}) = \begin{cases} i + 1 & \text{if } t \text{ is a closed term whose value is } i < \xi. \\ \xi & \text{otherwise} \end{cases}$
if \( s \) is a closed term whose value is \( i \) and \( t_1 \) is a closed term representing the same numeral as \( t_2 \).

Definition 3 (level \( lv(A) \) of formula \( A \) in the language of \( ID_1(U) \)) For the formula \( A \) in the language of \( ID_1(U) \), the level \( lv(A) \) of the formula \( A \) is defined inductively as follows:

- \( lv(P) := 0 \) for any atom of the language of \( PA \).
- \( lv(A \land B) := \max\{lv(A), lv(B)\} \)
- \( lv(\forall x.A) := \begin{cases} \max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \)
- \( lv(A \lor B) := \begin{cases} \max\{lv(A) + 1, lv(B)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \)
- \( lv(P^d t) := \begin{cases} 1 & \text{if } d(P^d t) = \xi \\ 0 & \text{otherwise} \end{cases} \)
- \( lv(t \prec s \land P^d t) := \begin{cases} 1 & \text{if } d(P^d t) = \xi \\ 0 & \text{otherwise} \end{cases} \)

The subsystems \( S_k(ID_1(U)) \) and \( S_k(ID\xi(U)) \) of \( ID_1(U) \) and \( ID\xi(U) \) are defined in terms of level \( lv \) as follows:

Definition 4 (the subsystem \( S_k(ID\xi(U)) \) of \( ID\xi(U) \)) \( S_k(ID\xi(U)) \) is \( ID\xi(U) \) except that for every abstract \( V \) in \( (A.2)_\xi \), \( (TI)_\xi \) and \( (VJ), lv(V) \leq k \) holds. \( S_k(ID\xi(U)) := \bigcup \{S_k(ID\xi(U) \mid U \text{ is a positive operator form} \} \)

The following notation is introduced:

Notation 1 Let \( TI[\alpha, \gamma, \mu] \) denote the schema defined as \( TI[\alpha, \gamma, \mu] := \alpha(\mu) \land (Prog[\alpha, \gamma, V] \land \forall \nu. (\gamma(\mu, \nu) \land \alpha(\nu) \land V(\nu))) \). And \( TI[\alpha, \gamma, \mu]_Q \) is the result of \( TI[\alpha, \gamma, \mu] \) by substituting \( Q \) for \( V \).

Notation 2 \( \omega(\xi, 0, \alpha) := \alpha \) and \( \omega(\xi, n + 1, \alpha) := (\xi, \omega(\xi, n, \alpha)) \).

Then by checking Arai's well ordering proof of \( O(\xi + 1, 1) \) [1] carefully, Proposition 1 is easily observed.

Proposition 1 For a formula \( Q \) with \( lv(Q) \leq 2 \) and \( k > 2 \), \( TI[F_0, <_0, \omega(\xi, k, 0)] \) is provable in \( S_k(ID\xi(U_0)) \). Namely, the ordinal \( \omega(\xi, k, 0) \) is accessible in \( S_k(ID\xi(U_0)) \) with respect to \( <_0 \).

Proof. We only consider the case in which \( \xi \) is a limit. (See Remark after Proposition 2 for the successor case.) Let \( \bigcup_{k < \xi} A_k := \{\mu \mid \forall k < \xi A_k(\mu)\} \). In Lemma 3 of [1] \( (TI)_\xi \) is used with the abstract \( i \) \( \forall \mu \ll < i(A_k(\mu)) \). In Lemma 3 of [1] \( (TI)_\xi \) is used with the abstract \( i \) \( \forall \mu \ll < i(A_k(\mu)) \). In Lemma 4 of [1] \( (A.2)_\xi \) is used with the abstract \( i \) \( A_k(\mu) \). In Lemma 4 of [1] \( (A.2)_\xi \) is used with the abstract \( i \) \( A_k(\mu) \). Then in Lemma 5 of [1] it is shown that \( TI[F_0, <_0, (\xi, 0)]_Q \) is provable in \( ID\xi(U_0) \) for each unary predicate \( Q(x) \) in \( ID\xi(U) \); In the case where \( \lim(\xi) \), \( (A.2)_\xi \) are used.
with the abstract \(\{x\} (x <_\xi (i, 0) \rightarrow Q(x))\) for all \(i < \xi\) (with level \(lv(Q)\)). In the case where \(\text{Suc}(\xi), (A.2)\xi\) is used with the abstract \(\{x\} (x <_\xi (\xi, 0) \rightarrow Q(x))\) (with level \(lv(Q)\)).

Hence until now it is observed that

\[(I) \quad S_{\max(3, lv(Q))}(ID_\xi(U_0)) \vdash TI[F_\xi, <_\xi, (\xi, 0)]Q.\]

From \((I)\) it is derived in the way familiar by Gentzen [8] that

\[(II) \quad S_{k+3}(ID_\xi(U_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, k + 3, 0)]Q\] with \(lv(Q) \leq 2\) and \(k \geq 0\).

Let us observe the proof of \((II)\). In Lemma 7 of [1] it is shown that \(\text{Prog}^{}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}^{}[F_\xi, <_\xi, s[Q]]\), where \(s[Q]\) is a jump operator defined as \(s[Q](\mu) := \forall \rho(F_\xi(\rho) \rightarrow \forall \psi <_\xi \rho(F_\xi(\psi) \rightarrow Q(\psi)) \rightarrow \forall \psi <_\xi \rho + ((\xi, \mu)^\xi(F_\xi(\psi) \rightarrow Q(\psi)))\), where \(\lambda \mu \mu + \nu^\xi\) is a primitive recursive function which is a generalization of \(\lambda \nu \mu \nu + \omega^\mu\) of Gentzen [8] and defined in [1] as follows:

- If \(\mu = 0\), then \(\mu + \nu^\xi = \nu + \mu^\xi = \nu\)
- Suppose \(\mu \neq 0\) and \(\nu \neq 0\) and
  \[\mu \equiv \mu_1 \& \cdots \& \mu_m \text{ with } \mu_1 \geq_\xi \cdots \geq_\xi \mu_m \neq 0\]
  \[\nu = \nu_1 \& \cdots \& \nu_n \text{ with } \nu_1 \geq_\xi \cdots \geq_\xi \nu_n \neq 0\]
  Let \(l\) be the number such that \(0 \leq l \leq m\) and \(\mu_l \leq_\xi \nu_1 <_\xi \mu_{l+1}\), then
  \[\mu + \nu^\xi := \mu_1 \& \cdots \& \mu_l \& \nu_{l+1} \& \cdots \& \nu_n\]

Note that \(lv(s^n[Q]) = n+\max(2, lv(Q))\) with \(n \geq 1\), where \(s^n[Q] := [s[\cdots[s[Q]\cdots]]]\)

Let us sketch the proof of \(\text{Prog}^{}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}^{}[F_\xi, <_\xi, s[Q]]\) due to Gentzen [8], where a mathematical induction of the level \(\leq lv(Q)\) is used;

Assume

\[\text{Prog}^{}[F_\xi, <_\xi, Q]\]

\[F_\xi(x) \land \forall y <_\xi x (F_\xi(y) \rightarrow s[Q](y))\]

We have to show \(s[Q](x)\). So assume further

\[F_\xi(\rho)\]

\[\forall \psi <_\xi \rho(\xi, f(\xi, \psi, \rho)) \rightarrow Q(\psi)\]

\[\forall \psi <_\xi \rho(\xi, f(\xi, \psi, \rho))^\xi \land F_\xi(\xi, f(\xi, \psi, \rho))\]

Under the above assumptions \((1) \sim (5)\), we have to show \(Q(\psi)\).

Consider the case where \(x \neq 0\). Since \(\nu <_\xi \rho \oplus (\xi, x)^\xi\), there exists primitive recursive functions \(f\) and \(g\) such that \(\nu <_\xi \rho \oplus (\xi, f(\xi, \psi, \rho)) \& g(\xi, \psi, \rho)\) with \(f(\xi, \nu, \rho) <_\xi x\) and \(F_\xi(f(\xi, \nu, \rho))\). From \((2)\), \(s[Q](f(\xi, \nu, \rho))\) holds. Then a universal instantiation with \(\rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n\) (note that \(\rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n <_\xi \rho \oplus (\xi, x)^\xi\)) for an arbitrary \(n\) allows the following:

\[F_\xi(\rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n(F_\xi(\xi, f(\xi, \nu, \rho))^\xi \& \xi \& F_\xi(\xi, f(\xi, \nu, \rho))^\xi \& n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow Q(\eta)\]

\[F_\xi(\rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n)\] (from \((5)\)) and the property of \(\text{Suc}\), the following holds;

\[\forall \eta <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi . \text{Suc}(\eta)(F_\xi(\eta) \rightarrow Q(\eta))\]

Then mathematical induction with abstract \(\{n\}(\forall \eta <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& n(F_\xi(\eta) \rightarrow Q(\eta)))\), whose level is \(\max(2, lv(Q))\), implies (with \((4)\)) \(\forall \eta <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& g(x, \nu, \rho)(F_\xi(\eta) \rightarrow Q(\eta))\). Hence from \((5)\), \(Q(\nu)\) holds.

Consider the case where \(x = 0\). For each formula \(Q\), \(s[Q]\) denotes the formula of the following form; \(s[Q](\mu) := \forall \psi(F_\xi(\rho) \rightarrow \forall \nu <_\xi \rho(F_\xi(\psi) \rightarrow Q(\psi)) \rightarrow \forall \nu <_\xi \rho \oplus (\xi, f(\xi, \nu, \rho))^\xi \& g(x, \nu, \rho)(F_\xi(\eta) \rightarrow Q(\eta))\). Then we can prove without \((A.1)_\xi, (A.2)_\xi, TI_\xi\) and the mathematical induction that \(\text{Prog}^{}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}^{}[F_\xi, <_\xi, s[Q]]\). As is shown
above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level $\leq \text{Max}(2, lv(Q))$.

From now we assume $lv(Q) \leq 2$. With the help of $\text{Prog}[F_{\xi}, <_{\xi}, Q] \rightarrow \text{Prog}[F_{\xi}, <_{\xi}, s(Q)]$ and $\text{Prog}[F_{\xi}, <_{\xi}, s(Q)] \rightarrow \text{Prog}[F_{\xi}, <_{\xi}, s^2(Q)]$, in which proof all mathematical inductions are restricted to those of level $\leq 3$, (I) implies the following $(II)_0$;

$$(II)_0 \quad S_0(ID_{\xi}^1(U_0)) \vdash TI[F_{\xi}, <_{\xi}, \omega(\xi, 3, 0)]_Q$$

By replying this method, the above $(II)$ is obtained.

Then following Arai [1], the next proposition is derived from $(II)$.

$S_{k+3}(ID_{\xi}^1(U_0)) \vdash TI[F_0, <_{0}, \omega(\xi, k + 3, 0)]_Q$ with $lv(Q) \leq 2$ and $k \geq 0$.

Hence the proposition holds.

Using the above, Proposition 2 follows;

**Proposition 2** For $k > 2$, the ordinal up to $\omega(\xi, k + 1, 0)$ is accessible in $S_k(ID_{\xi}^1(U_0))$ with respect to $<_{0}$.

**Remark 1:**

From the case in which $\xi$ is a successor ordinal, the transfinite induction formula \{i\} $\text{Prog}[F_{\xi}, <_{\xi}, \bigcap_{k < i} A_i]$ at the beginning of the proof of Proposition 1 above is replaced by \{i\} $\text{Prog}[F_{\xi}, <_{\xi}, A_i]$, which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for $k > 1$.

### 3 Unprovability of the transfinite induction up to $\omega(\xi, k + 1, 0)$ in system $S_k(Al_{\xi}^-)$

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

$$S_k(ID_{\xi}) \not\vdash TI[F_0, <_{0}, \omega(\xi, k + 1, 0)]$$ for $k > 2$

On the whole segment of $ID_{\xi} = \bigcup_n S_n(ID_{\xi})$, Arai [3] proves that $ID_{\xi} \not\vdash TI[F_0, <_{0}, O(\xi + 1, 1)]$. Note that $O(\xi + 1, 1) := \bigcup_{\omega(\xi, k, 0)}$. He shows that the consistency of $ID_{\xi}$ is provable using transfinite induction up to $O(\xi + 1, 1)$ by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

$$TI[F_0, <_{0}, \omega(\xi, k + 1, 0)] \vdash \text{Cons}(S_k(ID_{\xi}))$$ for $k > 2$

Our crucial point is to introduce a $n$-height $h_n$ for each $\eta \leq \xi$ (Definition 11) and consider an ordinal assignment to a proof $< P, \{h_n\}_{\eta \leq \xi, d}$ with $\xi$-sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system $Al_{\xi}^-$ of $\xi$-times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System $Al_{\xi}^-$ is defined by adding the following principles based on $PA$.  

Definition 5 (System $AI^-_\xi$, cf. Arai [3])
For any arithmetical form $B$, the following axioms schemata are added.

\[
(Q^B : \text{right}) \quad \Gamma \vdash \Delta, B(X, Q^B_{\xi t}, t, s)
\]

where $Q^B_{\xi t} := \{x, y\} (x \prec t \land Q^B xy)$

\[
(Q^B : \text{left}) \quad t \prec \xi, Q^B ts \rightarrow B(V, Q^B_{\xi t}, t, s)
\]

We assume that the language contains only $\forall, \neg$ and $\land$ for the logical connectives.

Then, the definition of $\nu$ in the previous section is modified as follows;

Definition 6 ($\eta$-level $\nu_\eta(A)$ of a formula $A$ with $\eta \preceq \xi$) For the formula $A$ in the language of $AI^-_\xi$ and an ordinal $\eta \preceq \xi$, the $\eta$-level $\nu_\eta(A)$ of the formula $A$ is defined inductively as follows, where $d$ is defined in Definition 2 of previous section with using $Q_8$ instead of $pU$ and $d(Xt) := 0$ (for $X$ a predicate variable):

\[
\nu_\eta(P) := 0 \text{ for any atom of } L_{PA}.
\]

\[
\nu_\eta(A \land B) := \max\{\nu_\eta(A), \nu_\eta(B)\}
\]

\[
\nu_\eta(\forall x A) := \begin{cases} 
\max\{2, \nu_\eta(A)\} & \text{if } \nu_\eta(A) \geq 1 \\
0 & \text{if } \nu_\eta(A) = 0
\end{cases}
\]

\[
\nu_\eta(\neg A) := \begin{cases} 
\nu_\eta(A) + 1 & \text{if } \nu_\eta(A) \geq 1 \\
0 & \text{if } \nu_\eta(A) = 0
\end{cases}
\]

\[
\nu_\eta(Q^B_{\xi}) := \begin{cases} 
1 & \text{if } d(Q^B_{\xi}) = \eta \\
0 & \text{otherwise}
\end{cases}
\]

\[
\nu_\eta(t \prec s \land Q^B_{\xi}) := \begin{cases} 
1 & \text{if } d(t \prec s \land Q^B_{\xi}) = \eta \\
0 & \text{otherwise}
\end{cases}
\]

We can define the fragments $S_k(AI^-_\xi)$ in the same manner as $S_k(ID_\xi)$ as follows.

Definition 7 (the subsystem $S_k(AI^-_\xi)$ of $AI^-_\xi$) $S_k(AI^-_\xi)$ is $AI^-_\xi$ except that for every abstract $V$ in $Q^B_{\xi}$:left and $(VJ)$, $\nu_\xi(V) \leq k$ holds.

$ID_\xi$ is obtained from $ID_\xi'$ in the previous section by changing the underlying logic from the intuitionistic to the classical.

For each formula $F$ of the language of $ID_\xi$, we define a formula $F^*$ of the language of $AI_\xi$ by substituting $Q^B_{\xi}$ for all occurrences of $P^\xi$, where

\[
B(X, Y, c_0, c_1) := \forall y(U(X, Y, c_0, y) \rightarrow X y) \rightarrow X c_1.
\]

It is well known that by this $*$, $ID_\xi$ is embeddable into $AI_\xi$ (cf. [3]). Obviously $\nu(F) = \nu_\xi(F^*)$ holds i.e., $\xi$-level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occurring in a proof figure of $AI^-_\xi$ are of the following normal form:

Lemma 1 (the normal form of a formula in $AI^-_\xi$) For arbitrary formula $A$ of the language of $AI^-_\xi$, there exists a formula of the following form, called a normal formula, which is equivalent to $A$ (in $LK$):

\[
\forall x_1 \ldots \forall x_n \forall y \exists y D[Q^B_{t_1 s_1}, \ldots, Q^B_{t_m s_m}]
\]

where $D[*_1, \ldots, *_m]$ is a context of the language of $PA$, and no quantifier occurring in $D$ bounds any $*_i$ ($1 \leq i \leq m$) and $\nu_\eta(D[Q^B_{t_1 s_1}, \ldots, Q^B_{t_m s_m}]) \leq 2$ for any $\eta \preceq \xi$. 

Definition 8 (normal proofs) Let $S$ be a sequent of normal formulas. A normal proof of $S$ is a proof in which $\forall$-left rules are used, instead of $\forall$-left rules in a proof:

$$
\Gamma \vdash \Delta, A(t_1, \ldots, t_n) \\
\forall x_1 \cdots x_n \neg A(x_1, \ldots, x_n), \Gamma \vdash \Delta
$$

Note that the original $\neg$-left rule may also appear in a normal proof.

Lemma 2 Any provable sequent of normal formulas has a normal proof.

From now on we assume any $S^\xi(A\xi^-)$-proof to be normal by virtue of the above two lemmata.

Definition 9 For each formula $A$, $\eta(A) \leq \xi$ is defined as $\eta(A) := \max\{\eta \mid lv_\eta(A) \neq 0\}$.

Definition 10 ($g_\eta(A)$ with $\eta < \xi$)

$$
g_\eta(A) := \begin{cases} 
g(A) & \text{if } \eta(A) \geq \eta \\
0 & \text{if } \eta(A) < \eta \end{cases}
$$

where $g(A)$ denotes the number of logical symbols in $A$.

We modify the notion of proof with degree $< P, d >$ of Arai [3] into $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$ by introducing $\xi$-sort of height $\{h_\eta\}_{\eta \leq \xi}$, as follows:

Definition 11 (A proof with $\xi$-sort of height $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$) A proof $\langle P, d \rangle$ (with degree $d$) is called a proof with $\xi$-sort of height $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$ if for each sequent $S$ of $P$ and each ordinal $\eta \leq \xi$, a natural number $h_\eta(S)$ satisfying the following condition is assigned. We call $h_\eta$ a $\eta$-height.

0. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $S$ is the end sequent of $P$.

For the last inference $I$ of the form

$$
I \quad \frac{S'}{S}
$$

1. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $I$ is a substitution.

2. $h_\eta(S) = h_\eta(S')$ for every $\eta \leq \xi$ if $I$ is an inference except substitution, induction and cut.

3. $\begin{cases} 
1 \quad h_\eta(S) \geq \max\{h_\eta(S'), g_\eta(D)\} & \text{for } \eta < \xi \\
2 \quad h_\xi(S) = \max\{h_\xi(S'), lv_\xi(D)\} & \text{if } I \text{ is a cut, where } D \text{ is the cut formula of the inference } I.
\end{cases}$

4. $\begin{cases} 
1 \quad h_\eta(S) \geq \max\{h_\eta(S'), g_\eta(D)\} + 1 & \text{for } \eta < \xi \\
2 \quad h_\xi(S) = \max\{h_\xi(S'), lv_\xi(D)\} + 1 & \text{if } I \text{ is an induction.}
\end{cases}$

Definition 12 For each sequent $S$ of $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$, $\eta(S) \leq \xi$ is defined as $\eta(S) := \begin{cases} 
d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\
\max\{\eta \mid h_\eta(S) \neq 0\} & \text{otherwise}
\end{cases}$

The following is an immediate consequence from Definition 12.
Lemma 3 For any proof with $\xi$-sort of height $< P, \{h_n\}_{n<\xi}, d >$ and for any inference $I$ (with a lower sequent $S'$ and an upper sequent $S$) in $< P, \{h_n\}_{n<\xi}, d >$,
\[ \eta(S) \geq \eta(S') \]
holds.

Notation 3 For $i \leq \xi$ and an ordinal diagram $\alpha$, an ordinal diagram $\omega(i, n, \alpha)$ is defined inductively as follows.
- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha))$

Definition 13 (ordinal assignment) Let $I$ be an inference of the form

\[
I \quad \frac{S_1}{S_2} \]

Then $O(S)$ is defined as follows:

1. When $I$ is a cut,
   \[ O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), \omega(\eta(S_1), h_{\eta(S_1)}(S_1), O(S_1) \# O(S_2))) \]
   Here $k := \text{Max}\{h_{\eta(T)}(T) \mid T \text{ is above } I\}$ and $\omega[*] := \omega(\gamma_1, k, \omega(\gamma_2, k_2, \ldots, \omega(\gamma_n, k_n, *))$),
   where $\{\gamma_1, \ldots, \gamma_n\} := \{\gamma \mid \eta(S) < \gamma < \eta(S_1) \text{ and } h_\gamma(T) \neq 0 \text{ for some } T \text{ above } I\}$
   with $\gamma_1 < \cdots < \gamma_n$ and $k_i := \text{Max}\{h_{\gamma_i}(T) \mid T \text{ is above } I\}$.

2. When $I$ is a logical inference,
   \[ O(S) := O(S_1) \# O(S_2) \# 0 \]

3. When $I$ is a structural inference,
   \[ O(S) := O(S_1) \# O(S_2) \]

4. When $I$ is a substitution,
   \[ O(S) := (d(I), O(S_1)) \]

Theorem 1 The transfinite induction on $\omega(\xi, k + 1, 0)$ is unprovable in $S_k(\Lambda I\xi^-)$ for $k > 2$.

Proof.
We refine the proof reduction process of Arai [3] to define the reduction process for $S_k(\Lambda I\xi^-)$ ($k > 2$), and show that the well-orderness of $\omega(\xi, k + 1, 0)$ implies the termination of the reduction process, hence the consistency of $S_k(\Lambda I\xi^-)$. Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)
Without loss of generality, we assume that all logical initial sequents of the form $p \rightarrow p$ where $p$ is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) dose work not only for a weakning in end-piece but also for a more general weakening with such a weakening formula $D$ as the bundle $T$ (cf. p78 of [14]) which begins with $D$ ends with a cut formula $D$ and no logical inference affect $T$.

\[ \text{In the case where } \eta(S) = \eta(S_1), \omega[*] \text{ is } * \text{ and } O(S) := \omega(\eta(S), k + h_{\eta(S)}(S_1) - h_{\eta(S)}(S), O(S_1) \# O(S_2)). \]
Then from sublemma 12.9 of [14], there exists a suitable cut $J$ in the end piece of $< P, \{h_\eta\}_{\eta \leq \xi}, d >$. Let $l_1$ and $l_2$ be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of $J$. We shall demonstrate following three essential cases both for limit ordinal $\xi$ and for successor ordinal $\xi$;

(Case 1) The case where the cut formula $C := A \land B$ with $\eta(C) < \xi$; Let $K$ (whose lower sequent is $T$ and whose upper sequent is $T_1$) denotes the upper-most inference below $J$ such that either (i) or (ii) holds:

$$\eta(T) = \eta(A) \land (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i)$$

$$\eta(T) < \eta(A) \cdots (ii)$$

where $A$ is the auxiliary formula of $l_1$ and $l_2$

$< P, \{h_\eta\}_{\eta \leq \xi}, d >$ is as follows:

$$\begin{array}{c}
S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

T_1 \\
U_1 \\
K \\

T_2 \\
U_2 \\
K \\

T^* \\

T \\

\end{array}$$

$< P', \{h'_\eta\}_{\eta \leq \xi}, d' >$ is as follows, where $l_1$ and $l_2$ are weakening-right (with a weakening formula $A_1$) and weakening-left (with a weakening formula $A_3$) respectively;

$$\begin{array}{c}
S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

S_1 \\
S_2 \\
S \\

T_1 \\
U_1 \\
K \\

T_2 \\
U_2 \\
K \\

T^* \\

T \\

\end{array}$$

(case 1.1): The case where (i) holds. Then for any sequent $T'$ between $S_1$ and $T$, $\eta(T') \geq \eta(A)$ holds.

(case 1.1.1) $\eta(T_1) = \eta(T)$

$O_{P'}(T^*) <_0 O_{P}(T)$ is checked as usual way.

(case 1.1.2) $\eta(T_1) > \eta(T)$

special case of (case 1.2)

(Case 1.2): The case where (ii) holds. Then $\eta(T) < \eta(A) \leq \eta(T_1)$ holds. We assign

$$h'_\eta(U_1) := \begin{cases} 
    g(A) & \text{if } \eta = \eta(A), \\
    0 & \text{otherwise}
\end{cases}$$

and $h'_\eta(T_{1^*}) := h'_\eta(T_1^*) := h_\eta(T_1)$ for all $\eta \leq \xi$.

Hence $\eta(U_1) = \eta(A)$ holds. On the other hand, there exist contexts $a$ and $b$ such
that $O_p(T) = \omega(\eta(T), k - h_{\eta(T)}(T), a[\omega(\eta(A), k, b[\alpha_1 \# \alpha_2])])$, 
$O_p(U_1) = \omega(\eta(U_1), m - h_{\eta(U_1)}(U_1), b[\alpha_1 \# \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha_1 \# \alpha_2])$ and 
$O_p(T^*) = \omega(\eta(T), k' - h_{\eta(T)}(T), a[\omega(\eta(U_1), g(A), O_p(U_1) \# O_p(U_2))])$
Since $\omega(\eta(A), k, b[\alpha_1 \# \alpha_2]) > \omega(\eta(U_1), g(A), O_p(U_1) \# O_p(U_2))$, $O_p(T^*) < O_p(T)$ holds.

(Case 2) The case where cut formula is $\forall \bar{x} \neg B(\bar{x})$: 
$< P, \{h_n\}_{n \leq i}, d >$ is as follows; here $I_2$ is $\forall$-left.

\[
\begin{array}{c}
S_{11} \quad S_{12} \quad I_1 \\
S_{21} \quad S_{22} \quad I_2 \\
S_{11} \quad S_{12} \quad J \\
\vdots \\
\vdots \\
\vdots \\
T' \quad K \\
\end{array}
\]

$< P', \{h_n'\}_{n \leq i}, d' >$ is as follows, where $I_1$ and $I_2$ are weakening-right and weakening-left (respectively) with weakening formulas $\forall \bar{x} \neg B(\bar{x})$. Note that by virtue of (preparation) and (elimination of weakening), any formula of the form $\neg B(\bar{x})$ which is an ancestor of the auxiliary formula of $I_1$ is a descendant of principal formulas of an inference $\neg$-right. Hence the following $S_{11}^T(\bar{x})$ can be obtained.

\[
\begin{array}{c}
S_{11}^I(t) \quad S_{12}^I(t) \quad I_1 \\
S_{21}^I(t) \quad S_{22}^I(t) \quad I_2 \\
S_{11}^I(e(t)) \quad S_{12}^I(e(t)) \quad J \\
\vdots \\
\vdots \\
\vdots \\
U_1(t) \quad K \\
\vdots \\
T' \quad U_2 \quad K \\
\end{array}
\]

Since $I_{v_n(B(\bar{x}))}(\forall \bar{x} \neg B(\bar{x})) > I_{v_n(B(\bar{x}))}(B(\bar{x}))$ holds, $O(P') < O(P)$ is checked as the usual way.

(Case 3) The case where the cut formula of $J$ is $Q^Bts$:

$< P, \{h_n\}_{n \leq i}, d >$ is as follows, where $K$ (with the lower sequent $T$) denotes the upper most inference below $J$ such that $\eta(T) \leq d(B(X, Q_{s_1}, t, s)) := i;$
Let $T_1$ denote such upper sequent of $K$ that is below $J$.

\[
\begin{array}{c}
S_{j_1}^1 \quad I_1 \quad S \\
S_{j_2}^j \quad J \\
T_1 \quad (T_2) \quad K
\end{array}
\]

$S : \quad t_2 < \xi, Q_{t_2} s_2 \rightarrow B(V, Q_{<t_2, t_2}, s_2)$

$S_i^j : \quad \Gamma_1 \rightarrow \Delta_1, \beta(X, Q_{<t_1, t_1}, t_1, s_1)$

$S_i^j : \quad \Gamma_1 \rightarrow \Delta_1, Q_{t_1} s_1$

$S_i^j : \quad \Gamma_2 \rightarrow \Delta_2, Q s$

$S_i^j : \quad Q s, \Pi \rightarrow \Lambda$

$S_i^j : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2$

$T_1 : \quad \Phi_1 \rightarrow \Psi_1$

$T_2 : \quad \Phi \rightarrow \Psi$

$O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), c[\omega(\eta(T_1), h_{\eta(T_1)}(T_1), O_P(T_1) \# O_P(T_2)])].$

$< P', \{h'_n\}_{n \leq \xi}, d' >$ is as follows. where $I_1$ is weakening-right with a weakening formula $Q^{\beta_{t_1, s_1}}$:

\[
\begin{array}{c}
S_{j_1}^1 \quad I_1 \quad S \\
S_{j_2}^j \quad J \\
T_1 \quad (T_2) \quad K
\end{array}
\]

We assign $\{h'_n\}_{n \leq \xi}$ as follows:

- $h'_n(T^*) := h_\eta(S)$ for all $\eta \leq \xi$
- $h'_n(T_1^*) := h_\eta(T_1)$ for all $\eta \leq \xi$.
- $h'_n(T^*) := \begin{cases} h_\eta(T) & \text{if } \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{cases}$

$O_P(T^*) = (i, O_P(T^*))$
\[ O_P(T^*) = \omega(\eta(T^*), l - h'_{n(T^*)}(T^*), d[\omega(\eta(T^*), h_n(T^*)]) O_P(T^*) \# O_P(T^*)) \]

Note that \( \eta(T^*) = i \). Obviously \( k = l \) from the figure of \( P' \). And from the above assignment \( h', c[*] = \omega(\gamma_1, k_1, \ldots, \omega(\gamma_s, k_s, d[*])) \) with \( \gamma_s < i = \gamma_{s+1} \). Hence \( O(P') < _0 O(P) \) holds.

The following Corollary is immediate from the above theorem and the fact that \( S_k(\mathcal{D}) \) is a subsystem of \( S_k(\mathcal{A}) \) under the interpretation * (cf. the paragraph after Definition 7).

**Corollary 1** The transfinite induction on \( \omega(\xi, k + 1, 0) \) is unprovable in \( S_k(\mathcal{D}) \) for \( k > 2 \).

**Proof.** As remarked after Definition 7, \( \xi \)-level does not change under the interpretation of an \( \mathcal{A} \)-formula to an \( \mathcal{D} \)-formula. Hence the Corollary is obvious.

**Theorem 2** (Main Theorem)

\[ |S_k(\mathcal{D})(U_0)| = |S_k(\mathcal{D})| = |S_k(\mathcal{A})| = |\omega(\xi, k + 1, 0)|_0 \text{ with } k > 2. \]

**Remark 2:** Our system \( S_k(\mathcal{D}) \) can be reformulated by means of the alternation complexity of quantifiers when we include \( \exists \) in our language. Here, a normal formula is of the form \( \exists t_1 \exists t_2 \ldots \exists t_m \mathcal{D}[P]\), where \( \mathcal{D}[*] \) is a context of the language of \( \mathcal{D} \) and with no quantifier occurring in \( \mathcal{D} \) bounds any \( \exists_i \) \( (1 \leq i \leq m) \). \( \{Q_j, \overline{Q}_j\} = \{\forall, \exists\} (j = 1, \ldots, m) \). \( lv \) is essentially the same as \( lv_\xi \) except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

\[ lv(\mathcal{D}[P]) := \left\{ \begin{array}{ll} 1 & \text{if all } P_i (i = 1, \ldots, m) \text{ is positive in } \mathcal{D} \\ 2 & \text{otherwise} \end{array} \right. \]

Then the \( lv \) of above normal formula is \( n+i \) if \( \overline{Q}_n = \forall \) and \( n+1+i \) if \( \overline{Q}_n = \exists \), where \( i := lv(\mathcal{D}[P]) \). \( S'_k(\mathcal{D}) \) is defined in the same way as the former definition of \( S_k(\mathcal{D}) \) with using the above new notation of \( lv \). It is easily seen that \( S'_k(\mathcal{D}) \) is equivalent to \( S_k(\mathcal{D}) \). In particular \( |S'_k(\mathcal{D})| = |\omega(\xi, k + 1, 0)|_0 \) with \( k > 2 \).

**References**


