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Values for Multialternative Games

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Abstract: We generalize the Banzhaf value in multialternative games defined by Bolger (1993). This value is produced by axioms similar to those of the Bolger value, just as the Banzhaf value produced by axioms similar to Shapley's.

1 Introduction

In characteristic function form simple games, players simply vote yea or nay. The multialternative game was first defined in Bolger (1993): the game contains more than two alternatives. Games allowing more than two coalitions have been also formulated as partition function games. Although the partition function game only refers to the formation of groups, the multialternative games makes clear which group of players chooses which alternative. They are very useful, especially to examine voting situations with three or more alternatives. Bolger defined a value based on a system of axioms, which is a generalization of the Shapley value (Shapley (1988)).

To evaluate voting situations, the Banzhaf value (Banzhaf (1965), Lehrer (1988)) is used as well as the Shapley value. Our purposes is to generalize the Banzhaf value in multialternative games. We will modify Bolger's axioms using the idea of Straffin (1976).

2 Multialternative Games and Bolger Value

2.1 Definition of Multialternative Games

In characteristic function form games, the characteristic function is given as $u : 2^N \to \Re$ ($u(\emptyset) = 0$), where 2^N is the set of all coalition $S \subset N = \{1, \cdots, n\}$, and $\Re$ is the set of real numbers. In multialternative games, we consider not only N but also the set of alternatives $A = \{a_1, \cdots, a_r\}$. The set S_j $(j = 1, \cdots, r)$ is the coalition consisting of players who choose a_j. The set S_j may be empty. The ordered coalition structure $P = (S_1, \cdots, S_r)$ is called an arrangement, where $\cup_{j=1}^{r} S_j = N$ and $S_j \cap S_{j'} = \emptyset$ if $j \neq j'$. Let j be fixed, and denote the set of pairs of S_j and an arrangement containing it by $\mathcal{P}^j$. That is,

$$\mathcal{P}^j = \{(S_j, P) | S_j \subset N, S_j \text{ is the } j\text{th element of } P\}.$$

For convenience, let

$$\mathcal{P} = \cup_{j=1}^{r} \mathcal{P}^j,$$

and

$$\mathcal{P}^j(i) = \{(S_j, P) \in \mathcal{P}^j | S_j \ni i\}.$$

We will also define the characteristic function just as in cooperative games. Although the characteristic function of cooperative games is defined with respect to a coalition, Bolger's characteristic function v is defined on $\mathcal{P}$, or $v : \mathcal{P} \rightarrow \Re$. The emptyset is always assumed to take 0, or $v(\emptyset, P) = 0$. The set of Bolger's characteristic functions is denoted by Γ. The triple (N, A, v), or v when players and alternatives are clear, is called a multialternative game.

Note that fuction v depends not only on S_j but P the jth component of which is S_j. When there are only two alternatives, one coalition S determines the other $N \setminus S$. In contrast, suppose a four-person voting game with three alternatives. Suppose the coalition S_j wins if and only if S_j contains more than the number of members in the other S_k's $(k \neq j)$. Formally, for each $j = 1, \cdots, r$,

$$v(S_j, P) = \begin{cases} 1, & \text{if } |S_j| > \max\{|S_k| : k \neq j\} \\ 0. & \text{otherwise} \end{cases}$$

Then $\{1, 2\}$ wins if 3 and 4 vote different candidates, but it loses if 3 and 4 vote the same candidate. Thus whether a coalition wins depends not only its size but also the size of other coalitions.

The partition function form game due to Lucas-Thrall (1963) also gives a similar generalization of the usual characteristic function. The partition function is defined with respect to a coalition S and a coalition structure $Q \ni S$[1]. Games of this form have been studied in many literatures. The Bolger's multialternative games is, however, basically different from the partition function form game. Suppose the example of the United Nations Security Council given in Bolger (1993). If most of the members in N vote yea (or nay), their claim will be adopted and the issue will be accepted (or rejected). But if they abstain, the result depends upon the votes of the other members. In this sense, alternatives *yea*, *nay* and *abstain* are not homogeneous. This is not captured by the partition function form representation. The Bolger's characteristic function, however, enables us to distinguish the situations where all the members choose yea, nay and abstain by giving three values,

$$v(N, (N, \emptyset, \emptyset)), v(N, (\emptyset, N, \emptyset)) \text{ and } v(N, (\emptyset, \emptyset, N)),$$

to these three situations.

2.2 Bolger Value

In the following, for convenience, fix one j and define a value for this particular j. Pick an arrangement P that has the jth element S_j. Suppose $i \in S_j$. Define α_{iS_k} as

$$\alpha_{iS_k}(P) = (S_1, \cdots, S_j \setminus \{i\}, \cdots, S_k \cup \{i\}, \cdots, S_r).$$

[1]Set $Q = \{S_1, \cdots, S_\ell\}$ is called a partition or a coalition structure, if $\cup_{j=1}^{\ell} S_j = N$ and $S_j \cap S_{j'} = \emptyset$ for $j \neq j'$. The partition function is defined on (S, Q), where $S \ni Q$. Thus the partition function for (S, Q) gives the quantity the members in S could gain if the coalition structure is Q.

This function moves player i from S_j to S_k in a given arrangement P.

Functions $\theta^j : \Gamma \to \Re^n$ are called value functions, where $\Re^n$ is the set of n-dimensional real vectors. Just as the Shapley value[2], the Bolger value $\theta^j(v)$ is derived from a system of axioms.

Axiom 1 (j-efficiency) *$\sum_{i\in N} \theta_i^j(v) = v(N;j)$, where $(N;j)$ means that all the players are in S_j; i.e.,*

$$(N;j) = (N, (\emptyset, \cdots, \emptyset, \underset{j}{N}, \emptyset, \cdots, \emptyset)).$$

This axiom, which Bolger did not treat as an axiom, corresponds to Shapley's efficiency axiom. The following three axioms correspond to Shaley's dummy player, additivity and symmetry axioms.

Axiom 2 (j-dummy) *Player i is a j-dummy in (N, A, v) if for all $(S_j, P) \in \mathcal{P}^j(i)$ and for all $k \neq j$,*

$$v(S_j, P) = v(S_j \setminus \{i\}, \alpha_{iS_k}(P)).$$

If player i is a j-dummy, then $\theta_i^j(v) = 0$.

Axiom 3 (linearity) *Pick two multialternative games (N, A, v) and (N, A, w) and a real number c, define the games $v + w$ and cv as*

$$(v + w)(S_j, P) = v(S_j, P) + w(S_j, P),$$

and

$$(cv)(S_j, P) = c \cdot v(S_j, P),$$

for each $(S_j, P) \in \mathcal{P}$. Then,

$$\theta^j(v + w) = \theta^j(v) + \theta^j(w),$$

and

$$\theta^j(cv) = c\theta^j(v).$$

[2]Shapley's axioms are as follows;

efficiency : $\sum_{i\in N} \varphi_i(u) = v(N)$

dummy player : if $u(S) = u(S \setminus \{i\})$ for all $S \ni i$, the value $\varphi_i(u) = 0$.

additivity : $\varphi(u) + \varphi(u') = \varphi(u + u')$

symmetry : Let $\pi : N \to N$ be a permutation; then, $\varphi_i(u) = \varphi_{\pi i}(\pi u)$.

Then the Shapley value, which is a unique value φ satisfying axioms above, is

$$\varphi_i(u) = \sum_{S\subset N, S\ni i} \frac{(s-1)!(n-s)!}{n!}[u(S) - u(S \setminus \{i\})].$$

Axiom 4 (symmetry) *Let $\pi : N \to N$ be a permutation, and define a game πv as*

$$\pi v(S_j, P) = v(\pi S_j, \pi P),$$

where $\pi P = (\pi S_1, \cdots, \pi S_r)$. Then, $\theta_i^j(v) = \theta_{\pi i}^j(\pi v)$.

Bolger added another axiom, called a pivot move axiom.

Axiom 5 (pivot move) *With respect to any two multialternative games (N, A, v) and (N, A, w), if for all $(S_j, P) \in \mathcal{P}^j(i)$,*

$$\sum_{k \neq j}[v(S_j, P) - v(S_j \setminus \{i\}, \alpha_{iS_k}(P))] = \sum_{k \neq j}[w(S_j, P) - w(S_j \setminus \{i\}, \alpha_{iS_k}(P))],$$

then $\theta_i^j(v) = \theta_i^j(w)$.

This last axiom considers that player i changes his choice from j to another option when he is in S_j, the jth element of P. It claims that the same value should be given in two games v and w if changes of v and w induced by changes of i's choice are the same in total. Suppose orders of the alternatives are different: v considers S_1, S_2 and S_3 as the groups of yea, nay and abstain, respectively; w considers them as yea, abstain and nay, respectively. Formally, for example, $v(S_1, (S_1, S_2, S_3)) = w(S_1, (S_1, S_3, S_2))$ holds. The pivot move axiom guarantees the value for v and for w should be the same.

Theorem 1 (Bolger (1993)) *For all (N, A, v), value function $\theta^j(v)$ satisfies Axioms 1–5, if and only if*

$$\theta_i^j(v) = \sum_{(S_j, P) \in \mathcal{P}^j(i)} \sum_{k \neq j} \frac{(s_j - 1)!(n - s_j)!}{n!(r-1)^{n-s_j+1}}[v(S_j, P) - v(S_j \setminus \{i\}, \alpha_{iS_k}(P))], \tag{1}$$

where $s_j = |S_j|$.

For the subsequent discussion, let us briefly review his proof. If part is easily shown. To show the only if part, construct basis games $\{v^{T_j,Q} | (T_j, Q) \in \mathcal{P}^j\}$ as follows;

$$v^{T_j,Q}(S_j, P) = \begin{cases} 1, & \text{if } S_j = T_j \text{ and } P = Q \\ 0. & \text{otherwise} \end{cases} \tag{2}$$

First, construct the value θ_i^j for $v^{T_j,Q}$, and then obtain the value for general v, using the equation

$$v = \sum_{(S_j, P) \in \mathcal{P}^j} v(S_j, P) v^{S_j, P}, \tag{3}$$

and the linearity axiom. Though the basis game $v^{T_j,Q}$ correspond to the unanimity game used in Shapley (1988), note that player $i \notin T_j$ is not a j-dummy player. In fact, he can make the losing coalition $T_j \cup \{i\}$ with respect to Q' win by changing his vote to k, where $\alpha_{iS_k}(Q') = Q$. To evaluate $\theta_i^j(v^{T_j,Q})$, Axiom 4 plays an important roll.

An intuitive interpretation can be given in the following manner. Fix the player $i \in S_j$, pick the other $(s_j - 1)$ members out of $N \setminus \{i\}$ arbitrary. Then, there are

$$\frac{(n-1)!}{(s_j-1)!(n-s_j)!}$$

possibilties of S_j. Any member out of S_j chooses one of the alternatives $A \setminus \{a_j\}$ with equal probability $1/(r-1)$. Then the probability that a (S_j, P) occurs is

$$\frac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-s_j}},$$

and there are $(r-1)$ patterns that move i to the other coalitions $S_k (k \neq j)$. Thus the coefficient is obtained. The Bolger value is considered to be a multialternative generalization of the wellknown Shapley value; it is clear that when $r = 2$, it gives the Shapley value.

3 A Banzhaf-type Value and Its Axiomatization

The axioms for the Banzhaf value was first given in Dubey-Shapley (1979), and later modified by Lehrer (1988). The Banzhaf index can be obtained by counting swings, players' combinations of yea and nay in which player i can change the final outcome. Dubey-Shapley used three indices related to Banzhaf index. One is the 'raw' Banzhaf index, which is simply the number of swings. Another one is normalized so that the sum of indices over players equals 1. This is just what Banzhaf (1965) originally obtained. The last one is called the swing probability, which is $1/2^{n-1}$ times 'raw' one. This is derived from the assumption that all the players except i vote yea and nay with equal probabilities. To obtain the 'raw' Banzhaf index, they used total swing axiom instead of Shapley's efficiency axiom, which claims the sum of the number of swings over players coincides with the sum of indices. Following their total swing axiom, we use the following axiom instead of Axiom 1.

Axiom 6 (total contribution) *Let define a contribution of player i by choosing a_j as*

$$\eta_i^j(v) = \sum_{(S_j,P)\in\mathcal{P}^j(i)} \sum_{k\neq j} [v(S_j, P) - v(S_j \setminus \{i\}, \alpha_{iS_k}(P))],$$

and a total contribution as

$$\bar{\eta}^j(v) = \sum_{i\in N} \eta_i^j(v).$$

Then,

$$\sum_{i\in N} \theta_i^j(v) = \bar{\eta}^j(v).$$

Before the theorem, let us define S-unanimity game v^S as follows;

$$v^S(S_j, P) = \begin{cases} 1, & \text{if } S_j \supset S \\ 0. & \text{otherwise} \end{cases}$$

This is the game that any coalition including S wins regardless of arrangements.

Theorem 2 *For all (N, A, v), value function $\theta^j(v)$ satisfies Axioms 2,3,4,5 and 6, if and only if*

$$\theta_i^j(v) = \sum_{(S_j,P)\in\mathcal{P}^j(i)} \sum_{k\neq j}[v(S_j, P) - v(S_j \setminus \{i\}, \alpha_{iS_k}(P))]. \tag{4}$$

Proof. The if part can be easily shown. To show the only if part, let $v^{S_j,P}$ and $i \in S$. Then, Axiom 6 implies $\bar{\eta}^j(v^{S_j,P}) = s_j(r-1)$, because each member in S_j can make the coalition lose by changing his choice k ($k \neq j$). Thus, $\theta_i^j(v^{S_j,P}) = r-1$ from Axiom 3.

When $i \notin S$, on the other hand, the inclusion-exclusion principle implies

$$\sum_{\substack{(S_j,P)\in\mathcal{P}^j(i)\\ S_j=S}} \theta_i^j(v^{S_j,P}) = \sum_{t=1}^{n-s}\left[(-1)^t \sum_{\substack{T\subset N\setminus S,|T|=t\\ T\ni i}} \theta_i^j(v^{S\cup T})\right]. \tag{5}$$

by Axioms 2 and 3. Note that we never have $t = 0$ since $i \notin S_j$. Axiom 5 implies

$$\sum_{\substack{(S_j,P)\in\mathcal{P}^j(i)\\ S_j=S}} \theta_i^j(v^{S_j,P}) = (r-1)^{n-s}\theta_i^j(v^{S_j,P}). \tag{6}$$

Since

$$\theta_i^j(v^S) = \begin{cases} r^{n-s}(r-1), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

from Axiom 6, equation (5) and (6) implies

$$(r-1)^{n-s}\theta_i^j(v^{S_j,P}) = (r-1)\sum_{t=1}^{n-s}(-1)^t \frac{t}{n-s}\binom{n-s}{t} r^{n-s-t}.$$

Using

$$\frac{t}{n-s}\binom{n-s}{t} = \binom{n-s-1}{t-1},$$

we obtain

$$\begin{aligned}(r-1)^{n-s}\theta_i^j(v^{S_j,P}) &= (r-1)\sum_{t=1}^{n-s}(-1)^t\binom{n-s-1}{t-1} r^{n-s-t} \\ &= -(r-1)\sum_{t=0}^{n-s-1}(-1)^t\binom{n-s-1}{t} r^{n-s-t-1} \\ &= -(r-1)^{n-s},\end{aligned}$$

which implies

$$\theta_i^j(v^{S_j,P}) = -1.$$

Consequently, the value for the basis game $v^{S_j,P}$ is summarized as

$$\theta_i^j(v^{S_j,P}) = \begin{cases} r-1, & \text{if } i \in S_j \\ -1, & \text{if } i \notin S_j \end{cases} \tag{7}$$

By the definition of the basis games (2) and Axiom 3,

$$\begin{aligned}
\theta_i^j(v) &= \sum_{(S_j,P)\in\mathcal{P}^j} v(S_j,P)\theta_i^j(v^{S_j,P}) \\
&= (r-1)\sum_{(S_j,P)\in\mathcal{P}^j(i)} v(S_j,P) + (-1)\sum_{(S_j,P)\in\mathcal{P}^j\setminus\mathcal{P}^j(i)} v(S_j,P) \\
&= (r-1)\sum_{(S_j,P)\in\mathcal{P}^j(i)} v(S_j,P) - \sum_{(S_j,P)\in\mathcal{P}^j(i)}\sum_{k\neq j} v(S_j\setminus\{i\},\alpha_{iS_k}(P)) \\
&= \sum_{(S_j,P)\in\mathcal{P}^j(i)}\sum_{k\neq j}[v(S_j,P) - v(S_j\setminus\{i\},\alpha_{iS_k}(P))].
\end{aligned}$$

Q.E.D.

Although this axiomatization corresponds to Bolger's axioms, Axiom 6 seems too self-induced. In fact, Dubey-Shapley's total swing axiom has been modified later by Lehrer (1988) and Straffin (1976). In the following section, we will present an axiomatization based on the idea of Straffin (1976).

4 Another Axiomatization

4.1 The Bolger Value

Straffin (1976) introduced a little different axioms from Shapley's. He used an axiom concerning unanimity games[3] instead of the efficiency axiom. We use the following axiom instead of Axioms 1 and 4.

Axiom 7 *For all i in S, $\theta_i^j(v^S) = 1/s$.*

Theorem 3 *For all (N,A,v), a value function $\theta^j(v)$ satisfies Axioms 2,3,5 and 7, if and only if it is given as (1).*

Before proceeding to the proof, let us prepare three lemmas.

Lemma 1

$$\begin{pmatrix} n \\ t \end{pmatrix}^{-1} - \begin{pmatrix} n+1 \\ t \end{pmatrix}^{-1} = \begin{pmatrix} n+1 \\ t+1 \end{pmatrix}^{-1} \frac{t}{t+1}. \tag{8}$$

[3]Define a unanimity game u^T as

$$u^T(S) = \begin{cases} 1, & \text{if } S \supset T \\ 0. & \text{otherwise} \end{cases}$$

Using the following axioms

unanimity game I :For any unanimity game u^T, the value $\varphi_i(u^T) = 1/|T|$ for all $i \in T$.

unanimity game II :For any unanimity game u^T, the value $\varphi_i(u^T) = 1/2^{|T|-1}$ for all $i \in T$.

instead of Shapley's efficiency and symmetry axioms also uniquely determines the Shapley value and the Banzhaf value, respectively.

Proof. trivial.

Lemma 2 *Take a sequence of real numbers* $\{A_t\}_{t=0}^n$ *arbitrary. Then,*

$$\sum_{t=0}^{n}(-1)^t \binom{n}{t} A_t = \sum_{t=0}^{n-1}(-1)^t \binom{n-1}{t} [A_t - A_{t+1}]. \tag{9}$$

Proof.

$$\begin{aligned}
&\sum_{t=0}^{n}(-1)^t \binom{n}{t} A_t \\
&= A_0 + \sum_{t=1}^{n-1}(-1)^t \left[\binom{n-1}{t} + \binom{n-1}{t-1}\right] A_t + (-1)^n A_n \\
&= \left[A_0 + \sum_{t=1}^{n-1}(-1)^t \binom{n-1}{t} A_t\right] + \left[\sum_{t=1}^{n-1}(-1)^t \binom{n-1}{t-1} A_t + (-1)^n A_n\right] \\
&= \sum_{t=0}^{n-1}(-1)^t \binom{n-1}{t} A_t + \sum_{t=0}^{n-1}(-1)^{t+1} \binom{n-1}{t} A_{t+1} \\
&= \sum_{t=0}^{n-1}(-1)^t \binom{n-1}{t} [A_t - A_{t+1}].
\end{aligned}$$

Q.E.D.

Lemma 3

$$\sum_{t=0}^{n-s}(-1)^t \binom{n-s}{t} \frac{1}{s+t} = \frac{1}{s}\binom{n}{s}^{-1}. \tag{10}$$

Proof. First, let

$$\frac{1}{s+t} = \binom{s+t}{1}^{-1},$$

be A_t in Lemma 2. By the use of Lemma 1 implies the left hand of (10) equals

$$\begin{aligned}
&\sum_{t=0}^{n-s-1}(-1)^t \binom{n-s-1}{t} \left[\binom{s+t}{1}^{-1} - \binom{s+t+1}{1}^{-1}\right] \\
&\quad = \sum_{t=0}^{n-s-1}(-1)^t \binom{n-s-1}{t} \frac{1}{2}\binom{s+t+1}{2}^{-1}.
\end{aligned} \tag{11}$$

Next, let

$$\binom{s+t+1}{2}^{-1},$$

be A_t in Lemma 2. Then, the preceding two lemmas similarly make (11) equal

$$\sum_{t=0}^{n-s-2}(-1)^t \binom{n-s-2}{t} \frac{1}{3}\binom{s+t+2}{3}^{-1}.$$

Repeatedly calculating it k $(1 \leq k \leq n-s)$ times in the same way yields

$$\sum_{t=0}^{n-s-k} (-1)^t \binom{n-s-k}{t} \frac{1}{k+1} \binom{s+t+k}{k+1}^{-1}.$$

Eventually when $k = n-s$, it equals

$$\begin{aligned} \frac{1}{n-s+1} \binom{n}{n-s+1}^{-1} &= \frac{(s-1)!(n-s+1)!}{n!(n-s+1)} \\ &= \frac{1}{s} \binom{n}{s}^{-1}. \end{aligned}$$

Q.E.D.

Proof of Theorem 3. Since it is clear that (1) satisfies Axioms 2,3,5 and 7, it is sufficient to show the only if part. Assume that θ_i^j satisfies these axioms. Let $S \subset N$ be fixed; then

$$\sum_{P \in \mathcal{P}^j, S_j = S} v^{S_j,P} = \sum_{t=0}^{n-s} \left[(-1)^t \sum_{T \subset N \setminus S, |T| = t} v^{S \cup T} \right]. \tag{12}$$

by the inclusion-exclusion principle. For $i \in S$, using Axioms 3 and 7, we obtain

$$\begin{aligned} \sum_{P \in \mathcal{P}^j, S_j = S} \theta_i^j(v^{S_j,P}) &= \sum_{t=0}^{n-s} \left[(-1)^t \sum_{T \subset N \setminus S, |T| = t} \theta_i^j(v^{S \cup T}) \right] \\ &= \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \frac{1}{s+t}. \end{aligned} \tag{13}$$

Further, since Axiom 5 implies each term in the left side takes the same value, fixing an (S_j, P) with $S_j = S$ makes the left side equal

$$(r-1)^{n-s} \theta_i^j(v^{S_j,P}).$$

Thus, from Lemma 3,

$$\begin{aligned} (r-1)^{n-s} \theta_i^j(v^{S_j,P}) &= \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \frac{1}{s+t} \\ &= \frac{1}{s} \binom{n}{s}^{-1}, \end{aligned}$$

which implies

$$\theta_i^j(v^{S_j,P}) = \frac{(s-1)!(n-s)!}{n!(r-1)^{n-s}}.$$

For $i \notin S$, on the other hand, Axioms 2, 3 and 5 imply (5) and (6). Thus, from Axiom 7,

$$(r-1)^{n-s} \theta_i^j(v^{S_j,P}) = \sum_{t=1}^{n-s} (-1)^t \frac{t}{n-s} \binom{n-s}{t} \frac{1}{s+t}.$$

Using

$$\frac{t}{n-s}\binom{n-s}{t}=\binom{n-s-1}{t-1},$$

we obtain

$$\begin{aligned}(r-1)^{n-s}\theta_i^j(v^{S_j,P}) &= \sum_{t=1}^{n-s}(-1)^t\binom{n-s-1}{t-1}\frac{1}{s+1}\\ &= -\sum_{t=0}^{n-s-1}(-1)^t\binom{n-s-1}{t}\frac{1}{s+t+1}\\ &= -\frac{1}{s-1}\binom{n}{s-1}^{-1},\end{aligned} \tag{14}$$

from Lemma 3, which implies

$$\theta_i^j(v^{S_j,P})=-\frac{(s-2)!(n-s+1)!}{n!(r-1)^{n-s}}.$$

The value for the basis game $v^{S_j,P}$ is summarized as

$$\theta_i^j(v^{S_j,P})=\begin{cases}\dfrac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-s_j}}, & \text{if } i\in S_j\\ -\dfrac{(s_j-2)!(n-s_j+1)!}{n!(r-1)^{n-s_j}}, & \text{if } i\notin S_j\end{cases}$$

By the definition of the basis games (2) and Axiom 3,

$$\begin{aligned}\theta_i^j(v) &= \sum_{(S_j,P)\in\mathcal{P}^j} v(S_j,P)\theta_i^j(v^{S_j,P})\\ &= \sum_{(S_j,P)\in\mathcal{P}^j(i)}\frac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-s_j}}v(S_j,P)+\sum_{(S_j,P)\in\mathcal{P}^j\setminus\mathcal{P}^j(i)}\frac{(s_j-2)!(n-s_j+1)!}{n!(r-1)^{n-s_j}}v(S_j,P)\\ &= \sum_{(S_j,P)\in\mathcal{P}^j(i)}\frac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-s_j}}v(S_j,P)-\sum_{(S_j,P)\in\mathcal{P}^j(i)}\sum_{k\neq j}\frac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-(s_j-1)}}v(S_j\setminus\{i\},\alpha_{iS_k}(P))\\ &= \sum_{(S_j,P)\in\mathcal{P}^j(i)}\sum_{k\neq j}\frac{(s_j-1)!(n-s_j)!}{n!(r-1)^{n-s_j+1}}[v(S_j,P)-v(S_j\setminus\{i\},\alpha_{iS_k}(P))].\end{aligned}$$

Q.E.D.

4.2 The Banzhaf-type Value

In this section, consider another value for multialternative games based on the Banzhaf value. We follow the next axiom given by Straffin (1976). Straffin changed the axiom concerning with unanimity games. Here, we follow his axioms.

Axiom 8 *For all i in S, $\theta_i^j(v^S) = 1/r^{s-1}$.*

To explain the difference between Axioms 7 and 8, he used the idea of the lottery that makes $i \in S$ a dictator or a dummy. Under the situation of Axiom 7, players in S face a lottery that makes a player a dictator with probability $1/s$ and a dummy with $1 - 1/s$. That is to say, exactly one out of s players can be a dictator. Under the situation of Axiom 8, on the other hand, players face a lottery that one be a dictator with $1/r^{s-1}$ and a dummy with $1 - 1/r^{s-1}$. In this case, player $i \in S$ can be a dictator only when all of the others in S agree. Assuming that they choose a_j with probability $1/r$, we obtain that i becomes a dictator with probability $1/r^{s-1}$.

Then a value is obtained following to Banzhaf's idea.

Theorem 4 *For all (N, A, v), $\theta^j(v)$ satisfies Axioms 2,3,5 and 8, if and only if*

$$\theta_i^j(v) = \frac{1}{r^{n-1}(r-1)} \sum_{(S_j,P)\in\mathcal{P}^j(i)} \sum_{k\neq j} [v(S_j, P) - v(S_j \setminus \{i\}, \alpha_{iS_k}(P))]. \tag{15}$$

Proof. The outline of the proof is analogous to that of Theorem 2. Since it is clear that (15) satisfies Axioms 2,3,5 and 8, it is also sufficient to show the only if part. Assume that θ_i^j satisfies these axioms, and let $S \subset N$ be fixed; then (12) follows. For $i \in S$, using Axioms 3 and 8, we obtain

$$\begin{aligned} \sum_{\substack{(S_j,P)\in\mathcal{P}^j(i)\\ S_j=S}} \theta_i^j(v^{S_j,P}) &= \sum_{t=0}^{n-s} \left[(-1)^t \sum_{T\subset N\setminus S, |T|=t} \theta_i^j(v^{S\cup T}) \right] \\ &= \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \frac{1}{r^{s+t-1}}. \end{aligned} \tag{16}$$

Since the left hand can be replaced by $(r-1)^{n-s}\theta_i^j(v^{S_j,P})$ by Axiom 5,

$$\begin{aligned} (r-1)^{n-s}\theta_i^j(v^{S_j,P}) &= \frac{1}{r^{n-1}} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} r^{n-s-t} \\ &= \frac{(r-1)^{n-s}}{r^{n-1}}, \end{aligned}$$

which implies

$$\theta_i^j(v^{S_j,P}) = \frac{1}{r^{n-1}}.$$

For $i \notin S$, on the other hand, equations (5) and (6) hold from Axioms 2,3 and 5. From Axiom 8,

$$(r-1)^{n-s}\theta_i^j(v^{S_j,P}) = \frac{1}{r^{n-1}} \sum_{t=1}^{n-s} (-1)^t \frac{t}{n-s} \binom{n-s}{t} r^{n-s-t}.$$

Using

$$\frac{t}{n-s} \binom{n-s}{t} = \binom{n-s-1}{t-1},$$

we obtain

$$
\begin{aligned}
(r-1)^{n-s}\theta_i^j(v^{S_j,P}) &= \frac{1}{r^{n-1}}\sum_{t=1}^{n-s}(-1)^t\begin{pmatrix} n-s-1 \\ t-1 \end{pmatrix} r^{n-s-t} \\
&= -\frac{1}{r^{n-1}}\sum_{t=0}^{n-s-1}(-1)^t\begin{pmatrix} n-s-1 \\ t \end{pmatrix} r^{n-s-t-1} \\
&= -\frac{(r-1)^{n-s-1}}{r^{n-1}},
\end{aligned}
$$

which implies

$$
\theta_i^j(v^{S_j,P}) = -\frac{1}{r^{n-1}(r-1)}.
$$

Consequently, the value for the basis game $v^{S_j,P}$ is summarized as

$$
\theta_i^j(v^{S_j,P}) = \begin{cases} \dfrac{1}{r^{n-1}}, & \text{if } i \in S_j \\ \dfrac{-1}{r^{n-1}(r-1)}, & \text{if } i \notin S_j \end{cases}
$$

Note that this value is $1/r^{n-1}(r-1)$ times (7). By the definition of the basis games (2) and Axiom 3, equation (15) holds. Q.E.D.

When $r = 2$, value $\theta_i^j(v)$ gives the swing probability. Thus, this $\theta_i^j(v)$ is a generalization of it. An interpretation of this value is given in the following way. If all the players except i vote yea or nay with equal probabilities 1/2, a combination of yea and nay occurs with probability $1/2^{n-1}$. Similarly, if they choose one of r alternatives with equal probabilities, a combination of their choices occurs with $1/r^{n-1}$. Here, player i also has $(r-1)$ options; that is, he may change his choice from j to one of the other $(r-1)$ alternatives. Thus, this value is also a swing probability where player i and his option j are fixed.

5 Concluding Remarks

Multialternative games are useful to treat voting situations with more than two alternatives. We have proposed a multialternative generalization of the Banzhaf value.

We first modified Bolger (1993)'s axioms. Replacing the j-efficiency axiom by the total contribution axiom, just as in Dubey-Shapley (1977), yielded a generalized Banzhaf value, which we called the Banzhaf-type value.

But the total contribution axiom seems too self-induced. Thus, we use another axiomatization following Straffin (1976). There are two kinds of axioms with respect to unanimity games. One is that all the members should allocate the total value equally. Using this instead of the j-efficiency and symmetry axioms yields the Bolger value. The other is based on the situation that all the members choose an alternative with equal probability. Using this instead of the symmetry and the total contribution axiom yields the Banzhaf-type value.

We have followed the axiomatizations of Dubey-Shapley (1977) and Straffin (1976). Lehrer (1988) presented another axiomatization. He used an axiom called superadditivity, which is a kind of consistecy. An axiomatization following it will be done in the future.

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