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Linear operators and c.o.n.s. in Hilbert spaces associated with chaotic dynamical systems

(カオス力学系に付随する線形作用素および完全正規直交系)

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In this note, we will show a non-chaotic property in chaotic dynamical system (X, φ), where X is a compact set and φ is a continuous map of X onto X . One of the most important property in the chaotic dynamical theory is to be sensitive dependence on initial conditions, whichi is a chaotic property in the sense that there exists $\delta > 0$ such that, for any x in X and neighbourhood $U(x)$ of x, there exists y in $U(x)$ and $n \geq 0$ such that

$$d(\varphi^n(x), \varphi^n(y)) > \delta.$$

On the other hand, some chaotic dynamical systems have the property of toplogical-mixing on a measure spoce (X, m), that is,

$$\lim_{n\to\infty} \int_X f(\varphi^n(x))g(x)dm = \int_X f(x)dm$$

for any continuous function f on a metric space X and L^1- function g on the measurable space (X, m) with $\int_X g(x)dm = 1$.

We study this non-chaotic property by representing chaotic dynamical systems (X, φ) on Hilbert spaces $\mathfrak{H}$.

1. Covariant representation of dynamical systems

Let $C(X)$ be the C^*-algebra of all continuous functions on X. Then a continuous map φ from X onto itself induces a $*$-endomorphism α_φ of $C(X)$, which is defined by

$$\alpha_\varphi(f)(x) = f(\varphi(x)), \quad x \in X.$$

Let π be a covariant representation of $(C(X), \alpha_\varphi)$ on a Hilbert space $\mathfrak{H}$ in the following sense:

$$\pi(\alpha_\varphi(f)) = V_1\pi(f)V_1^* + V_2\pi(f)V_2^*$$

for all f in $C(X)$, where (V_1, V_2) is a couple of isometries on $\mathfrak{H}$ with the property

$$V_1V_1^* + V_2V_2^* = I.$$

In this case, (V_1, V_2) induces a $*$-endomorphism of $\mathfrak{L}(\mathfrak{H})$ as follows:

$$\alpha_V(a) = V_1 a V_1^* + V_2 a V_2^*$$

for all a in $\mathfrak{L}(\mathfrak{H})$.

For some couples (V_1, V_2), we can find a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ satisfying following condition:

$$V_1 e_n = e_{2n-1} \quad \text{and} \quad V_2 e_n = e_{2n} \quad \text{for all } n \geq 1,$$

which are called a c.o.n.s. of Walsh type with respect to (V_1, V_2).

Related to these systems, we have already had a theorem which shows a non-chaotic property.

Theorem 1.1. [2:Theorem 2.2.3] *Let π be a covariant representation of $(C(X), \alpha_\varphi)$ implememted by (V_1, V_2). If (V_1, V_2) has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, then we have*

$$\lim_{n\to\infty} (\alpha_V^n(a)\xi, \xi) = (ae_1, e_1)$$

for all a in A and ξ in $\mathfrak{H}$ with $\| \xi \| = 1$.

Here we give some examples. Let φ be a unimodal map of $[0,1]$ onto itself in the following sense.

(1) φ is a continuous map of $[0,1]$ onto $[0,1]$.

(2) There exists a point c in $(0,1)$ such that

  (i) $\varphi(0) = \varphi(1) = 0$ and $\varphi(c) = 1$,

  (ii) φ is strictly monotone increasing on $[0,c]$ and strictry monotone decreasing on $[c,1]$,

  (iii) φ and the two inverse maps β, γ of φ are abosolutely continuous functions on $[0,1]$, where $\beta([0,1]) = [0,c]$ and $\gamma([0,1]) = [c,1]$.

Given a unimodal map φ, we define a couple (V_1, V_2) of isometries associated with φ as follows:

$$V_1 = V_1(\varphi) = M_{\sqrt{\varphi'}} M_{\chi_{[0,c]}} T_\varphi \quad \text{and} \quad V_2 = V_2(\varphi) = -M_{\sqrt{-\varphi'}} M_{\chi_{[c,1]}} T_\varphi,$$

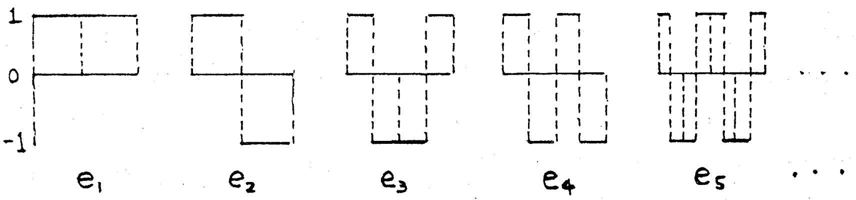
where M_f means the multiplication operator on $L^2[0,1]$, χ_E the characteristic function of E and $(T_\varphi \xi)(x) = \xi(\varphi(x))$.

Let $\pi(f) = M_f$ for f in $C(X)$. Then π is a covariant representation of $(C(X), \alpha_\varphi)$ with respect to this couple (V_1, V_2), but it has no c.o.n.s. of Walsh type. However, in the following example, we can find a couple (W_1, W_2) of isometris which implements the $*$-endomorphism α_V and has a c.o.n.s of Walsh type.

Example 1.2. Let $X = [0,1]$ and $\varphi = 1- \mid 1-2x \mid$: the tent map.

$$W_1 = \frac{1}{\sqrt{2}}V_1 - \frac{1}{\sqrt{2}}V_2 (= T_\tau) \quad \text{and} \quad W_2 = W_2(\tau) = \frac{1}{\sqrt{2}}V_1 + \frac{1}{\sqrt{2}}V_2.$$

Then π is also a covariant representation of $(C([0,1]), \alpha_\varphi)$ implemented by (W_1, W_2) having a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, which is just the following Walsh series (cf.[3]).



Putting $a = M_f$ for f in $C(X)$ in the theorem above, we have

$$\lim_{n\to\infty} (\alpha_\varphi^n(f)\xi, \xi) = \lim_{n\to\infty} (\alpha_V^n(f)\xi, \xi) = (fe_1, e_1) = (f, e_1)$$

for all ξ in $\mathfrak{H}$ with $\| \xi \| = 1$.

Remark.1.3. The Walsh series becomes a group with respect to product of fuctoins on [0,1], which satisfies the following relation.

Group $G = \{e_n\}_{n=1}^{\infty}$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	$\cdots$
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	
e_2	e_2	e_1	e_4	e_3	e_6	e_5	e_8	e_7	
e_3	e_3	e_4	e_1	e_2	e_7	e_8	e_5	e_6	
e_4	e_4	e_3	e_2	e_1	e_8	e_7	e_6	e_5	
e_5	e_5	e_6	e_7	e_8	e_1	e_2	e_3	e_4	
e_6	e_6	e_5	e_8	e_7	e_2	e_1	e_3	e_4	
e_7	e_7	e_8	e_5	e_6	e_3	e_4	e_1	e_2	
e_8	e_8	e_7	e_6	e_5	e_4	e_3	e_2	e_1	
$\vdots$									

Let φ be topologically conjugate to the tent map τ, that is, $\varphi = h \circ \tau \circ h^{-1}$ for some homeomorphism h of [0,1] onto itself. In our case, the maps h and h^{-1} are assumed to be absolutely continuous functions on [0,1].

Then $(C(X), \alpha_\varphi)$ has a covariant representation implemented by (W_1, W_2) defined as in Example 1.2. The couple (W_1, W_2) has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type with $e_1 = \sqrt{(h^{-1})'}$, where $(h^{-1})'$ is the derivative of h^{-1}.

Example 1.4. Let $X = [0,1]$ and $\varphi = 4x(1-x)$: the logistic map. Then φ is topologically cojugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$. Thus the couple (W_1, W_2) has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type with $e_1(x) = 1/(\pi(x(1-x))^{1/2})^{1/2}$.

2. Convergence of sequences $\{\ (\alpha_V^n(\cdot)\xi, \xi)\ \}_{n=1}^{\infty}$ in σ-weak topology

We consider the convergence in Theorem 1.1 in the context of duality between $\mathfrak{L}(\mathfrak{H})$ and the predual space $\mathfrak{L}(\mathfrak{H})_*$. Let M be an α_V-invariant von Neuman subalgebra of $\mathfrak{L}(\mathfrak{H})$ and $A = A_V^M$ the adjoint operator of the restriction of α_V to M. Then Theorem 1.1 implies the following. If (V_1, V_2) has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, then we have

$$\lim_{n\to\infty} A^n(\omega_{\xi,\xi}) = \omega_{e_1,e_1}$$

for all ξ in $\mathfrak{H}$ with $\| \xi \| = 1$, where $\omega_{\xi,\xi}$ is a vector state on $\mathfrak{L}(\mathfrak{H})$.

Let M be the abelian von Neumann subalgebra $M_{L^\infty[0,1]}$. Then the predual M_* is regarded as $L^1[0,1]$. Let (W_1, W_2) be as in Example 1.2 or 1.4. Then we have $A = A_V^M = A_W^M$ and the convergence mentioned above means the following.

If φ is the tent map on [0,1], we have

$$\lim_{n\to\infty} A^n(\eta) = e_1^2 = e_1$$

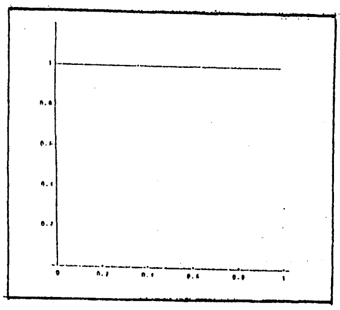
for all η in $L^1[0,1]$ with $\| \eta \|_1 = 1$, where $e_1(x) = 1$.

On the other hand, if φ is the logistic map on [0,1], we have.

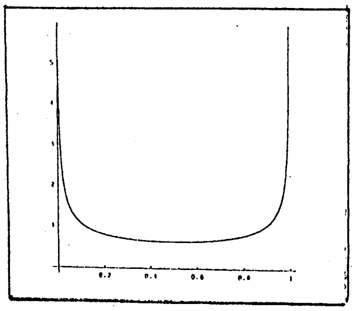
$$\lim_{n\to\infty} A^n(\eta) = e_1^2$$

for all η in $L^1[0,1]$ with $\| \eta \|_1 = 1$, where $e_1^2(x) = 1/\pi\sqrt{x(1-x)}$.

$e_1^2(x) = 1$



$e_1^2(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$



In the case of the tent map, we have more detailed results mentioned below.

Theorem 2.1 *Let φ be the tent map on* [0,1] *and M the abelian von Neumann algebra* $M_{L^\infty[0,1]}$. *Put $A = A_V^M$. Then we have the following.*

(1) $A(\eta)(x) = \frac{1}{2}(\eta(\frac{x}{2}) + \eta(1-\frac{x}{2}))$ *for η in $L^1[0,1]$.*

(2) $\lim_{n\to\infty} A^n(\eta) = 1$ *in $\sigma(L^1[0,1], L^\infty[0,1])$ topology, for η in $L^1[0,1]$ with $\| \eta \|_1 = 1$.*

(3) $\lim_{n\to\infty} \| A^n(\eta) - 1 \|_1 = 0$ *for η in $L^2[0,1]$ with $\| \eta \|_1 = 1$.*

(4) $\lim_{n\to\infty} \| A^n(\eta) - 1 \|_\infty = 0$ *for η in $C_W[0,1]$, where $C_W[0,1]$ is the C^*-subalgebra of $L^\infty[0,1]$ generated by the Walish series $\{e_n\}_{n=1}^\infty$.*

We note that

$$C[0,1] \subset C_W[0,1] \subset L^\infty[0,1] \subset L^2[0,1] \subset L^1[0,1].$$

Remark 2.2. Let A be the map of $L^1[0,1]$ into $L^1[0,1]$ mentioned in the theorem above. Then we have the following.
(1)$A(1) = 1$.
(2)$A(2x) = 1$.
(3)$A^n(3x^2) = \frac{3}{4^n} - \frac{3}{4^{n-1}\cdot 2} + \frac{4^{n-1}\cdot 2 + 1}{4^{n-1}\cdot 2}$ for each positive integer n.

For other α_V-invariant von Neuman subalgebras M, we have some results concerning the property of convergence of the sequence $\{ (A_V^M)^n(\omega_{\xi,\xi}) \}_{n=1}^\infty$. Moreover we are studying representations of chaotic dynamical systems in the case of other unimodal maps and Dai and Larson [1] provide an interesting theory which is deeply related to our study.

References

[1] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets* to appear as Memoire of A.M.S.

[2] S. Kawamura, Covarinat representations chaotic dynamical systems, to appear in Tokyo J. Math.

[3] F. Schipp, W.R. Wade and P. Simon, *Walsh series,* Adam Hilger, Bristol-New York, 1990.