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Statistical Estimation of Pure States

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Abstract

A statistical parameter estimation theory for quantum pure state models is presented. First, we formulate the one-parameter estimation theory in an analogous way to that of strictly positive models, and clarify the differences between them. We next investigate the multi-parameter estimation theory based on the right logarithmic derivatives.

1 Introduction

A quantum statistical model is a family of density operators ρ_θ defined on a certain separable Hilbert space $\mathcal{H}$ with finite-dimensional real parameters $\theta = (\theta^i)_{i=1}^n$ which are to be estimated statistically. In order to avoid singularities, the conventional quantum estimation theory [1][2] has been often restricted to models that are composed of strictly positive density operators. It was Helstrom [3] who successfully introduced the symmetrized logarithmic derivative for the one-parameter estimation theory as a quantum counterpart of the logarithmic derivative in the classical estimation theory. The right logarithmic derivative is another successful counterpart introduced by Yuen and Lax [4] in the expectation parameter estimation theory for quantum gaussian models, which provided a theoretical background of optical communication theory. Quantum information theorists have also kept away from degenerated states, such as pure states, for mathematical convenience [5]. Indeed, the von Neumann entropy cannot distinguish the pure states, and the relative entropies often diverge.

In this paper, however, we try to construct an estimation theory for pure state models, and clarify the differences between the pure state case and the strictly positive state case. First, we formulate the one-parameter pure state estimation theory, which seems quite analogous to the strictly positive case, but reveals the features of the pure state estimation theory. We next investigate the multi parameter estimation theory. Even in the strictly positive case, there is no general theory for multi parameter quantum estimation as yet. So, we restrict ourselves here to the theory based on the right logarithmic derivatives. All the results are presented without proofs. They will be found in future publications.

2 Review of the conventional theory

We first give a brief summary of the conventional quantum parameter estimation theory. Let

$$\mathcal{S} = \{\rho_\theta \; ; \; \rho_\theta = \rho_\theta^* > 0, \; \mathrm{Tr}\, \rho_\theta = 1, \; \theta \in \Theta \subset \mathbf{R}^n\} \tag{1}$$

be the statistical parametric model composed of strictly positive density operators. Here, θ is the parameter to be estimated statistically. Let $M(d\theta) = M(d\theta^1 \cdots d\theta^n)$ be a generalized measurement [1][2] which takes values on Θ. The corresponding probability distribution at the state ρ_θ is $P_\theta^M(B) = \mathrm{Tr}\, \rho_\theta M(B)$, where B is a Borel set on Θ. In the following, we identify the estimator for θ with the measurement on Θ. The expectation vector with respect to the measurement M at the state ρ_θ is defined as

$$E_\theta[M] = \int \hat{\theta} P_\theta^M(d\hat{\theta}).$$

The measurement M is called unbiased if $E_\theta[M] = \theta$ holds for all $\theta \in \Theta$, i.e.,

$$\int \hat{\theta}^j P_\theta^M(d\hat{\theta}) = \theta^j, \quad (j = 1, \cdots, n). \tag{2}$$

Differentiation yields

$$\int \hat{\theta}^j \frac{\partial}{\partial \theta^k} P_\theta^M(d\hat{\theta}) = \delta_k^j, \quad (j, k = 1, \cdots, n). \tag{3}$$

If (2) and (3) hold at a certain θ, M is called locally unbiased at θ. Obviously, M is unbiased iff M is locally unbiased at every $\theta \in \Theta$. Letting M be a locally unbiased measurement at θ, we define the covariance matrix $V_\theta[M] = [v_\theta^{jk}] \in \mathbf{R}^{n \times n}$ with respect to M at the state ρ_θ by

$$v_\theta^{jk} = \int (\hat{\theta}^j - \theta^j)(\hat{\theta}^k - \theta^k) P_\theta^M(d\hat{\theta}). \tag{4}$$

In order to obtain lower bounds for $V_\theta[M]$, let us consider a quantum analogue of the logarithmic derivative denoted by L_θ:

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \frac{1}{2}[\rho_\theta L_{\theta,j} + L_{\theta,j}^* \rho_\theta]. \tag{5}$$

For instance,

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \frac{1}{2}[\rho_\theta L_{\theta,j}^S + L_{\theta,j}^S \rho_\theta], \quad L_{\theta,j}^S = L_{\theta,j}^{S*} \tag{6}$$

defines the symmetrized logarithmic derivative (SLD) $L_{\theta,j}^S$ introduced by Helstrom [3], and

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \rho_\theta L_{\theta,j}^R \tag{7}$$

defines the right logarithmic derivative (RLD) $L_{\theta,j}^R$ introduced by Yuen and Lax [4]. Thus, (5) defines a certain class of logarithmic derivatives. Correspondingly, we define the quantum analogue of Fisher information matrix $J_\theta = [(L_{\theta,j}, L_{\theta,k})_{\rho_\theta}]$, where the inner product $(\cdot,\cdot)_\rho$ is defined by

$$(A, B)_\rho = \mathrm{Tr}\, \rho\, BA^*. \tag{8}$$

Then, the following quantum version of Cramér–Rao theorem holds.

Theorem 2.1 *For any locally unbiased measurement M, the following inequality holds:*

$$V_\theta[M] \geq (\mathrm{Re}\, J_\theta)^{-1}, \tag{9}$$

where $\mathrm{Re}\, J_\theta = (J_\theta + \overline{J_\theta})/2$. In particular, for the SLD, $J_\theta^S = \mathrm{Re}\, J_\theta = \left[\mathrm{Re}\, (L_{\theta,j}^S, L_{\theta,k}^S)_{\rho_\theta}\right]$ is called the SLD-Fisher information matrix. Moreover, for the RLD,

$$V_\theta[M] \geq (J_\theta^R)^{-1} \tag{10}$$

holds, where $J_\theta^R = \left[(L_{\theta,j}^R, L_{\theta,k}^R)_{\rho_\theta}\right]$ is called the RLD-Fisher information matrix.

When the model is one dimensional, the inequalities in the theorem become scalar. In this case, it can be shown that the lower bound $(\mathrm{Re}\, J_\theta)^{-1} = (J_\theta)^{-1}$ becomes most informative, i.e., it takes the maximal value, iff the SLD is adopted, and the corresponding lower bound $(J_\theta^S)^{-1} = 1/\mathrm{Tr}\, \rho_\theta (L_\theta^S)^2$ can be attained by the estimator $T = \theta I + L_\theta^S / J_\theta^S$, where I is the identity. Thus, the one-parameter quantum estimation theory is quite analogous to the classical one when the SLD is used.

On the other hand, for the dimension $n \geq 2$, the matrix equalities in (9) and (10) cannot be attained in general, because of the impossibility of the simultaneous measurement of non-commutative observables. We must, therefore, abandon the strategy of finding the measurement that minimizes the covariance matrix itself. Rather, we often adopt another strategy as follows: Given a positive definite real matrix $G = [g_{jk}] \in \mathbf{R}^{n\times n}$, find the measurement M that minimizes the quantity

$$\mathrm{tr}\, GV_\theta[M] = \sum_{jk} g_{jk} v_\theta^{jk}. \tag{11}$$

If there is a constant C such that $\mathrm{tr}\, GV_\theta[M] \geq C$ holds for all M, C is called a Cramér–Rao type bound or simply a CR bound, which may depend on both G and θ. For instance, it can be shown that the folowing two quantities are both CR bounds [6].

$$\begin{aligned} C^S &= \mathrm{tr}\, G(J_\theta^S)^{-1}, \\ C^R &= \mathrm{tr}\, G\,\mathrm{Re}\,(J_\theta^R)^{-1} + \mathrm{tr}\ \mathrm{abs}\, G\,\mathrm{Im}\,(J_\theta^R)^{-1}. \end{aligned}$$

Here $\mathrm{Im}\, A = (A - \overline{A})/2i$ and $\mathrm{tr}\ \mathrm{abs}\, A$ denotes the absolute sum of the eigenvalues of A. These CR bounds are called, respectively, the SLD–bound and the RLD–bound. The most informative CR bound is the maximum value of such C for given G and θ. Yuen and Lax [4] proved that the above C^R is the most informative for the gaussian model, and they explicitly constructed the optimum measurement which attains C^R. Holevo [2] derived another CR bound which, though an implicit form, is not less informative than C^S and C^R. Nagaoka [6] investigated in detail the relation between these CR bounds. He also derived a new CR bound for 2 dimensional models, which is not less informative than Holevo's one, and obtained explicitly the most informative CR bound specific to the spin 1/2 model. The construction of the general quantum parameter estimation theory for $n \geq 2$ is left to future study.

3 One-paremeter pure state model estimation theory

In this section, we give an estimation theory for one-parameter pure state models :

$$\mathcal{S} = \{\rho_\theta \,;\, \rho_\theta^2 = \rho_\theta,\ \mathrm{Tr}\, \rho_\theta = 1,\ \theta \in \Theta \subset \mathbf{R}\}. \tag{12}$$

Let $\mathcal{L}$ and $\mathcal{L}_{sa}$ are, respectively, the set of all the (bounded) linear operators and all the self-adjoint operators on $\mathcal{H}$. An unbiased estimator for θ is a self-adjoint operator T such that Tr $\rho_\theta T = \theta$ holds for all θ. The SLD for the model $\mathcal{S}$ is also defined by (6). In this case, however, SLD is not uniquely determined. Denote the set of all the SLD's at θ by $\mathcal{T}^S(\rho_\theta)$. Furthermore, let us define a pre-inner product on $\mathcal{L}$ by

$$\langle A, B\rangle_\rho = \frac{1}{2}\mathrm{Tr}\,\rho\,[BA^* + A^*B], \quad A, B \in \mathcal{L},$$

which depends on the state ρ. Note the sesquilinear form $\langle\cdot,\cdot\rangle_\rho$ becomes an inner product on $\mathcal{L}$ iff $\rho > 0$, and $\langle A, B\rangle_\rho = \mathrm{Re}(A, B)_\rho$ holds when $A, B \in \mathcal{L}_{sa}$. Denote by $\mathcal{K}_{sa}(\rho)$ the set of such self-adjoint operators K satisfying $\langle K, K\rangle_\rho = 0$. The following theorems show that the one-parameter estimation theory for the pure state models is quite analogous to that for the strictly positive models.

Theorem 3.1 *The SLD-Fisher information $J_\theta^S = \langle L_\theta^S, L_\theta^S\rangle_{\rho_\theta}$ is uniquely determined on the SLD tangent space $\mathcal{T}^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$, and is identical to*

$$J_\theta^S = 4\,\mathrm{Tr}\,\rho_\theta\left(\frac{d\rho_\theta}{d\theta}\right)^2. \tag{13}$$

Theorem 3.2 *For any unbiased estimator T, the following quantum Cramér–Rao inequality holds:*

$$V_\theta[T] \geq \frac{1}{J_\theta^S}. \tag{14}$$

The equality at θ holds iff

$$T = \theta I + \frac{2}{J_\theta^S}\frac{d\rho_\theta}{d\theta} + K_\theta, \quad {}^\forall K_\theta \in \mathcal{K}_{sa}(\rho_\theta). \tag{15}$$

Since $d\rho_\theta/d\theta$ and K_θ do not commute in general, the measurement which attains the lower bound (14) is not determined uniquely. This fact provides significant features in the pure state estimation theory.

Example 3.1 *Let us consider a model of the form*

$$\rho_\theta = e^{i\theta\mathcal{H}/\hbar}\rho_0\, e^{-i\theta\mathcal{H}/\hbar}.$$

Here, $\mathcal{H}$ is the time independent Hamiltonian of the system, $\hbar$ the Planck's constant, and θ the time parameter. Now, $L_\theta = -2i\mathcal{H}/\hbar$ is a logarithmic derivative which belongs to (5) *and the corresponding Cramér–Rao inequality (9) becomes*

$$V_\theta[T] \geq \frac{\hbar^2}{4V_\theta[\mathcal{H}]}, \qquad (16)$$

where T is an arbitrary unbiased estimator T for the time parameter θ. This inequality is nothing but a time-energy uncertainty relation. If $\rho_0 > 0$, then the general theory mentioned in section 2 says that this lower bound cannot be attained for any T since L_θ is not an SLD. On the other hand, if ρ_0 is pure, then the SLD–Fisher information (13) becomes $J_\theta^S = 4V_\theta[\mathcal{H}]/\hbar^2$, and the corresponding Cramér–Rao inequality (14) is identical to (16). Moreover, Theorem 3.2 asserts that the equality can be attained locally. This is a significant difference between the strictly positive models and the pure state models. Since both the logarithmic derivative $L_\theta = -2i\mathcal{H}/\hbar$ and the SLD–Fisher information (13) can be obtained directly from the Liouville–von Neumann equation, this result is not specific to the case where the Hamiltonian is time independent, but is quite general.

Example 3.2 *An unbiased estimator T is called efficient if the equality in (14) holds for all $\theta \in \Theta$. Let us consider a model of the form*

$$\rho_\theta = e^{if(\theta)A}\rho_0\, e^{-if(\theta)A},$$

where $f(\theta)$ is a real monotonic odd function and $A \in \mathcal{L}_{sa}$. If $\rho_0 > 0$, then it can be shown that there exists an efficient estimator for θ only when A is a canonical observable [7]. On the other hand, if ρ_0 is pure, then there may exist an efficient estimator even if A is not canonical, because of the uncertainty $K_\theta \in \mathcal{K}_{sa}(\rho_\theta)$ in (15). For instance, the spin $1/2$ model

$$f(\theta) = \frac{1}{2}\left(\frac{\pi}{2} - \cos^{-1}\theta\right), \quad A = \sigma_y, \quad \rho_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has an efficient estimator σ_z for the parameter θ.

4 Multi-parameter pure state model estimation theory

As was mentioned in section 2, there is no prototype for general theory of quantum multi-parameter estimation theory. So, let us restrict ourselves here to seeking the estimation theory based on the RLD. Since the RLD is defined by (7), it does not exist for degenerated states. However, what we need is not the RLD itself but the inverse of the RLD–Fisher information matrix, as is understood by (10). Following Holevo [2], we define the commutation operator $\mathfrak{D}$ on $\mathcal{L}_{sa}$ by

$$i(A\rho - \rho A) = \frac{1}{2}\left((\mathfrak{D}A)\rho + \rho(\mathfrak{D}A)\right), \qquad A \in \mathcal{L}_{sa}. \tag{17}$$

Note that $\mathfrak{D}A$ is uniquely determined for the given ρ and A iff $\rho > 0$.

Lemma 4.1 *If ρ is pure, then $\mathfrak{D}A$ is determined except for an uncertainty of the element of $\mathcal{K}_{sa}(\rho)$.*

From this lemma, $\mathfrak{D}$ can be regarded as a super-operator on $\mathcal{L}_{sa}/\mathcal{K}_{sa}(\rho)$. The following theorem gives the counterpart of the Holevo's result which was originally obtained in the strictly positive case [2, p. 280].

Theorem 4.1 *Suppose we are given a pure state model ρ_θ. Let $\{\rho_\theta(\varepsilon)\ ;\ \varepsilon > 0\}$ be a family of strictly positive density operators $\rho_\theta(\varepsilon)$ having a parameter ε which satisfy $\lim_{\varepsilon\downarrow 0}\rho_\theta(\varepsilon) = \rho_0$, and denote the corresponding RLD by $L_\theta^R(\varepsilon)$. If the SLD–tangent space $T^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$ is $\mathfrak{D}$-invariant, then*

$$\lim_{\varepsilon\downarrow 0}\left(J^R(\varepsilon)\right)^{-1} = \left(J^S\right)^{-1} + \frac{i}{2}\left(J^S\right)^{-1} D \left(J^S\right)^{-1}$$

holds, where $J^R(\varepsilon) = \left[(L_j^R(\varepsilon), L_k^R(\varepsilon))_{\rho_\theta(\varepsilon)}\right]$, $J^S = \left[\langle L_j^S, L_k^S\rangle_{\rho_\theta}\right]$, and $D = \left[i \operatorname{Tr} \rho_\theta[L_j^S, L_k^S]\right]$.

From this theorem, the inverse of the RLD–Fisher information matrix can be calculated directly from SLD, without using the diverging RLD–Fisher information matrix itself. Then, it may be important to investigate the condition for the SLD–tangent space to be $\mathfrak{D}$-invariant. The following theorem characterizes the structure of $\mathfrak{D}$-invariant SLD–tangent space.

Theorem 4.2 *The $\mathfrak{D}$-invariant SLD–tangent space $T^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$ has an even dimension and is decomposed into direct sum of 2 dimensional $\mathfrak{D}$-invariant subspaces. Moreover, by taking an appropriate basis of $T^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$, the operation of $\mathfrak{D}$ can be written in the* *form*

$$\mathfrak{D}\begin{bmatrix} \tilde{L}_1^S \\ \tilde{L}_2^S \\ \tilde{L}_3^S \\ \tilde{L}_4^S \\ \vdots \\ \tilde{L}_{2m-1}^S \\ \tilde{L}_{2m}^S \end{bmatrix} = \begin{bmatrix} 0 & 2 & & & & & \\ -2 & 0 & & & & & \\ & & 0 & 2 & & & \\ & & -2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 2 \\ & & & & & -2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{L}_1^S \\ \tilde{L}_2^S \\ \tilde{L}_3^S \\ \tilde{L}_4^S \\ \vdots \\ \tilde{L}_{2m-1}^S \\ \tilde{L}_{2m}^S \end{bmatrix}. \tag{18}$$

Definition 4.1 *The basis* $\{\tilde{L}_j^S\}_{j=1}^{2m}$ *of the SLD-tangent space* $T^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$ *which is subject to the transformation law (18) is called* ρ_θ*-symplectic.*

From Theorem 4.2, it is sufficient to consider a 2-dimensional $\mathfrak{D}$-invariant SLD tangent space. Indeed, the RLD–bound is decomposed as $C^R = \sum_{j=1}^m C_j^R$, where C_j^R is the RLD–bound with respect to the 2 dimensional subspace $\mathrm{span}\{\tilde{L}_{2j-1}^S, \tilde{L}_{2j}^S\}$ and is given by

$$C_j^R = C_j^S + \frac{\sqrt{\det G_j}}{\det J_j^S} \left|\mathrm{Tr}\, \rho_\theta[\tilde{L}_{2j-1}^S, \tilde{L}_{2j}^S]\right|. \tag{19}$$

Here, the subscript j denotes the restriction of the corresponding quantities onto the above 2-dimensional subspace. The following theorem gives the condition for the model to have a 2-dimensional $\mathfrak{D}$-invariant SLD–tangent space at ρ_θ.

Theorem 4.3 *For the pure state model* $\{\rho_\theta = |\theta\rangle\langle\theta|\}$*, the following two conditions are equivalent.*

(i) $\{\tilde{L}_j^S\}_{j=1,2}$ *is a* $|\theta\rangle\langle\theta|$*-symplectic basis.*

(ii) $(\tilde{L}_1^S + i\tilde{L}_2^S)|\theta\rangle = 0.$

The linear span of such basis $\mathrm{span}\{\tilde{L}_1^S, \tilde{L}_2^S\}$ *is* $\mathfrak{D}$*-invariant.*

The condition (ii) is nothing but the definition of the coherent states in quantum theory. Thus the $\mathfrak{D}$-invariancy is equivalent to the coherency of the model. Furthermore, the next theorem characterizes a global structure.

Theorem 4.4 *Consider the pure state model of the form $\rho_\theta = U_\theta \rho_0 U_\theta^*$ where $\{U_\theta\}$ forms a unitary group. This model has $\mathfrak{D}$-invariant SLD-tangent space for all θ iff $T^S(\rho_0)/\mathcal{K}_{sa}(\rho_0)$ is $\mathfrak{D}$-invariant, i.e., the model has a ρ_0-symplectic basis. Indeed, if $\{\tilde{L}_j^S\}_{j=1,2}$ is a ρ_0-symplectic basis, then $\{U_\theta \tilde{L}_j^S U_\theta^*\}_{j=1,2}$ becomes a ρ_θ-symplectic basis.*

Example 4.1 *Let us consider the family of canonical coherent states $\rho_z = |z\rangle\langle z|$ in a one dimensional harmonic oscillator with frequency ω, where $z = (\omega q + ip)/2\hbar \in \mathbf{C}$. This can be regarded as a 2-parameter pure state model which has real parameters q and p. It can be shown that the representative elements of SLD are*

$$L_q^S = \frac{2\omega}{\hbar}(Q - q), \qquad L_p^S = \frac{2}{\hbar\omega}(P - p),$$

and

$$\mathfrak{D}L_q^S = 2\omega L_p^S, \qquad \mathfrak{D}L_p^S = -\frac{2}{\omega}L_q^S.$$

Letting

$$\tilde{L}_q^S = \frac{\hbar}{2}L_q^S = \omega(Q - q), \qquad \tilde{L}_p^S = \frac{\hbar\omega}{2}L_p^S = P - p,$$

we have

$$\mathfrak{D}\tilde{L}_q^S = 2\tilde{L}_p^S, \qquad \mathfrak{D}\tilde{L}_p^S = -2\tilde{L}_q^S.$$

This indicates that $\{\tilde{L}_q^S, \tilde{L}_p^S\}$ forms a ρ_z-symplectic basis. Therefore, from Theorem 4.3,

$$(\tilde{L}_q^S + i\tilde{L}_p^S)|z\rangle = [\omega(Q - q) + i(P - p)]|z\rangle = 0,$$

which is nothing but the definition of canonical coherent states. Furthermore, from Theorem 4.1,

$$(J^R)^{-1} = \begin{bmatrix} \sigma_P^2 & i\hbar/2 \\ -i\hbar/2 & \sigma_Q^2 \end{bmatrix},$$

where $\sigma_P^2 = \hbar\omega/2$, $\sigma_Q^2 = \hbar/2\omega$, and the corresponding RLD-bound

$$g_P V_P[M] + g_Q V_Q[M] \geq g_P \sigma_P^2 + g_Q \sigma_Q^2 + \hbar\sqrt{g_P g_Q}$$

is identical to the pure state limit of the most informative CR bound obtained by Yuen and Lax [4].

Example 4.2 *We next show that the 2 dimensional spin coherent state model has $\mathfrak{D}$-invariant SLD-tangent space at every point. Let (θ, φ) be the polar coordinates where the north pole is $\theta = 0$ and x-axis corresponds to $\varphi = 0$. The spin coherent state $|\theta, \varphi\rangle$ is defined as*

$$|\theta, \varphi\rangle = R[\theta, \varphi]|j\rangle = \exp\left[i\theta(J_x \sin\varphi - J_y \cos\varphi)\right]|j\rangle,$$

where $|j\rangle$ is the highest occupied state in the spin j system. It can be shown that the SLD at the north pole in the direction of $\varphi = 0$ and $\varphi = \pi/2$ are, respectively, $2J_x$, $2J_y$ and the operation of $\mathfrak{D}$ becomes $\mathfrak{D}J_x = 2J_y$, $\mathfrak{D}J_y = -2J_x$. Therefore, $\tilde{L}_1^S = J_x$ and $\tilde{L}_2^S = J_y$ form a $|j\rangle\langle j|$-symplectic basis and

$$\left(\tilde{L}_1^S + i\tilde{L}_2^S\right)|j\rangle = J_+|j\rangle = 0,$$

where $J_+ = J_x + iJ_y$ is the spin creation operator. This is nothing but the definition of the terminal state $|j\rangle$. From this fact, we can immediately conclude that the model which comprises the totality of the spin coherent states

$$\rho_{\theta,\varphi} = |\theta, \varphi\rangle\langle\theta, \varphi| = R[\theta, \varphi]|j\rangle\langle j|R[\theta, \varphi]^{-1}$$

has $\mathfrak{D}$-invariant SLD tangent space at every point on the sphere. Indeed, since $R[\theta, \varphi]$ forms a compact Lie group, Theorem 4.4 asserts that

$$\left\{R[\theta, \varphi]\tilde{L}_1^S R[\theta, \varphi]^{-1},\ R[\theta, \varphi]\tilde{L}_2^S R[\theta, \varphi]^{-1}\right\}$$

forms a $|\theta, \varphi\rangle\langle\theta, \varphi|$-symplectic basis. Especially, a 2 parameter spin 1/2 model has $\mathfrak{D}$-invariant SLD tangent space and

$$\left(J^R\right)^{-1} = \frac{1}{\sin^2\theta}\begin{bmatrix} \sin^2\theta & -i\sin\theta \\ i\sin\theta & 1 \end{bmatrix}.$$

The corresponding CR bound is

$$g_\theta V_\theta[M] + g_\varphi V_\varphi[M] \geq g_\theta + \frac{g_\varphi}{\sin^2\theta} + \frac{2}{\sin\theta}\sqrt{g_\theta g_\varphi}.$$

This bound is the most informative CR bound for the model because it is identical to the pure state limit of the most informative CR bound obtained by Nagaoka [6].

5 Conclusions

A quantum estimation theory of the pure state models was presented. First, we constructed the one-parameter pure state estimation theory, which seems quite analogous to the strictly positive case, but discloses the characteristics of the pure state estimation theory. We next investigated the multi-parameter estimation theory, based on the right logarithmic derivatives. The construction of the general quantum multi-parameter estimation theory is left to future study, as is the strictly positive model case.

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