数理解析研究所講究録
980 巻 1997 年 137-149

Generalization of Integral Kernel Operators

NOBUAKI OBATA

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE
NAGOYA UNIVERSITY
NAGOYA, 464-01 JAPAN

Introduction

In most literatures creation and annihilation operators in a Fock space are introduced as operator-valued distributions though used in actual computation as if they were defined pointwisely. On the other hand, it is also possible to give a rigorous definition of such field operators at a point using a Gelfand triple or a rigged Hilbert space, see e.g., [1], [2]. The so-called white noise calculus initiated by Hida [3] offers one of such possibilities.

The foundation of white noise calculus is a Schwartz type distribution theory on a Gaussian space (E^*, μ); more precisely, it is based on a particular choice of a Gelfand triple:

$$(E) \subset L^2(E^*, \mu) \subset (E)^*,$$

where $L^2(E^*, \mu)$ is isomorphic to a Boson Fock space through the Wiener-Itô-Segal isomorphism. Then a pointwisely defined annihilation operator, which is also called Hida's differential operator and is denoted by ∂_t, becomes a continuous operator on (E); and a pointwisely defined creation operator ∂_t^* is a continuous operator on $(E)^*$.

In a series of works [10]–[12] we have established a systematic theory of operators on Gaussian space in terms of white noise calculus. The key role has been played by an *integral kernel operator* of which formal integral expression is given as

$$\int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \tag{1}$$

where κ is a distribution in $l+m$ variables. It should be emphasized strongly that an integral kernel κ can be a distribution. In fact, the composition $\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}$ is well defined (namely, normally ordered product) and becomes a continuous operator from (E) into $(E)^*$. Moreover, the dependence of the parameters s_j and t_k is smooth enough.

The kernel distribution κ in (1) being regarded as a scalar operator-valued distributon, we are led quite naturally to a generalization with an integral kernel being an operator-valued distribution. In this note we shall introduce an operator in the following form:

$$\int_{T^{l+m}} \partial_{s_1}^* \cdots \partial_{s_l}^* L(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m. \tag{2}$$

Of course, this is a formal (but sometimes very descriptive) expression. For the precise definition we need the characterization theorem for operator symbols and some properties of operator-valued distributions. Those results are obtained in [10]–[12].

As application we discuss an operator-valued (or quantum) stochastic process of Hitsuda-Skorokhod type. We shall observe that the classical case discussed in [4] (see also [7], [8]) is recovered as multiplication operator-valued processes. Our discussion is closely related to quantum stochastic calculus, in particular, to representation of quantum martingales, see [6], [9], [13], [14]. Further detailed study in this direction will appear in a forthcoming paper.

ACKNOWLEDGEMENTS. I am very grateful to Professors I. Kubo and H. Watanabe for interesting discussion and comments.

1 White noise functionals

We employ the standard setup for white noise calculus ([5], [10]–[12]) with the same notation as used there. Let T be a topological space with a Borel measure $\nu(dt) = dt$ which is thought of as a time parameter space when it is an interval, or more generally as a field parameter space. Given a positive selfadjoint operator A on the real Hilbert space $H = L^2(T, \nu; \mathbb{R})$ with Hilbert-Schmidt inverse, one may form a Gelfand triple:

$$E \subset H = L^2(T, \nu; \mathbb{R}) \subset E^*$$

in the standard manner; namely, E is the C^∞-domain of A equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0, \qquad \xi \in H, \quad p \in \mathbb{R},$$

where $|\cdot|_0$ is the norm of H. Such a countably Hilbert space is called a standard CH-space, see [11]. Since A^{-1} is of Hilbert-Schmidt type, E becomes a nuclear space. The canonical bilinear form on $E^* \times E$ and the real inner product of H are denoted by the same symbol $\langle \cdot, \cdot \rangle$ without contradiction.

One can think of E and E^* as spaces of test and generalized functions on T, respectively. In order to keep the delta functions δ_t within our discussion we assume:

(H1) for each $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T such that $\xi(t) = \tilde{\xi}(t)$ for ν-a.e. $t \in T$;

(H2) for each $t \in T$ a linear functional $\delta_t : \xi \mapsto \tilde{\xi}(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$;

(H3) the map $t \mapsto \delta_t \in E^*$, $t \in T$, is continuous with respect to the strong dual topology of E^*.

From now on we always assume that every element in E is a continuous function on T and do not use the symbol $\tilde{\xi}$. For another reason we need one more assumption:

(S) $\inf \mathrm{Spec}\,(A) > 1$.

We then put

$$\delta = \| A^{-1} \|_{\mathrm{HS}} < \infty, \qquad \rho = \| A^{-1} \|_{\mathrm{OP}} = (\inf \mathrm{Spec}\,(A))^{-1}.$$

The obvious inequalities

$$0 < \rho < 1; \qquad |\xi|_p \le \rho^q |\xi|_{p+q}, \qquad \xi \in E, \quad p \in \mathbb{R}, \quad q \ge 0,$$

are used throughout with no special notice.

The *Gaussian measure* μ is by definition a probability measure on E^* of which characteristic function is:

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx), \qquad \xi \in E.$$

The probability space (E^*, μ) is called a *Gaussian space*. We put

$$(L^2) = L^2(E^*, \mu; \mathbb{C})$$

for simplicity.

The canonical bilinear form on $(E^{\otimes n})^* \times (E^{\otimes n})$ is denoted by $\langle \cdot, \cdot \rangle$ again and its $\mathbb{C}$-bilinear extension to $(E_{\mathbb{C}}^{\otimes n})^* \times (E_{\mathbb{C}}^{\otimes n})$ is also denoted by the same symbol. For $x \in E^*$ let $:x^{\otimes n}:$ be defined as a unique element in $(E^{\otimes n})^*_{\mathrm{sym}}$ satisfying

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad \xi \in E_{\mathbb{C}}. \tag{3}$$

This "normalized" exponential function ϕ_ξ is called an *exponential vector*. In particular, ϕ_0 is the *vacuum*. As is well known, each $\phi \in (L^2)$ is expressed in the following form:

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad x \in E^*, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{4}$$

where each $x \mapsto \langle :x^{\otimes n}:, f_n\rangle$ and the convergence of the series are understood in the L^2-sense. Expression (4) is referred to as the *Wiener-Itô expansion* of ϕ. In that case,

$$\| \phi \|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, | f_n |_0^2 . \tag{5}$$

Thus we have a unitary isomorphism between (L^2) and the Boson Fock space over $H_{\mathbb{C}}$, which is the celebrated Wiener-Itô-Segal isomorphism.

The second quantized operator of A, denoted by $\Gamma(A)$, is an operator in (L^2) defined by

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, A^{\otimes n} f_n \right\rangle,$$

where $\phi \in (L^2)$ is given as in (4). Equipped with the maximal domain, $\Gamma(A)$ becomes a positive selfadjoint operator on (L^2) and we obtain a standard CH-space which will be denoted by (E). That $\Gamma(A)$ admits a Hilbert-Schmidt inverse is guaranteed by hypothesis (S). Therefore, (E) becomes a nuclear Fréchet space and we come to a complex Gelfand triple:

$$(E) \subset (L^2) = L^2(E^*, \mu; \mathbb{C}) \subset (E)^*.$$

Elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively. We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$ and by $\| \cdot \|_p$ the norm introduced from $\Gamma(A)$, namely,

$$\| \phi \|_p^2 = \| \Gamma(A)^p \phi \|_0^2 = \sum_{n=0}^{\infty} n! | (A^{\otimes n})^p f_n |_0^2 = \sum_{n=0}^{\infty} n! \, | f_n |_p^2 , \tag{6}$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related as in (4). Thus (5) is a special case of (6). As is easily seen from (6), $\phi \in (L^2)$ belongs to (E) if and only if $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$ for all n and $\sum_{n=0}^{\infty} n! \, | f_n |_p^2 < \infty$ for all $p \geq 0$.

We use a similar (but formal) expression for a generalized white noise functional. Every $\Phi \in (E)^*$ is written as

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \, F_n \right\rangle, \tag{7}$$

where $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}}$ and

$$\| \Phi \|_{-p}^2 = \sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 . \tag{8}$$

By construction $\| \Phi \|_{-p} < \infty$ for some $p \geq 0$, and hence for all sufficiently large $p \geq 0$. Expression (7) is also called the *Wiener-Itô expansion* of Φ. In that case,

$$\langle\!\langle \Phi, \, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, \, f_n \rangle ,$$

where $\phi \in (E)$ and its Wiener-Itô expansion is given as in (4).

2 Integral kernel operators

For any $y \in E^*$ and $\phi \in (E)$ we put

$$D_y \phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta y) - \phi(x)}{\theta}, \qquad x \in E^*. \tag{9}$$

It is known that the limit always exists and that $D_y \in \mathcal{L}((E),(E))$. Since the delta functions δ_t are elements in E^* by hypotheses (H1)–(H3), we may define

$$\partial_t = D_{\delta_t}, \qquad t \in T.$$

This is called *Hida's differential operator.* Obviously, ∂_t is a rigorously defined *annihilation operator* at a point $t \in T$. It should be therefore emphasized that ∂_t is *not* an operator-valued distribution but a continuous operator for itself. The *creation operator* is by definition the adjoint $\partial_t^* \in \mathcal{L}((E)^*, (E)^*)$ and we come to the so-called canonical commutation relation:

$$[\partial_s, \partial_t] = 0, \qquad [\partial_s^*, \partial_t^*] = 0, \qquad [\partial_s, \partial_t^*] = \delta_s(t) I, \qquad s,t \in T. \tag{10}$$

The last relation is understood in a generalized sense.

For $\phi, \psi \in (E)$ let $\eta_{\phi,\psi}$ be a function on T^{l+m} defined by

$$\eta_{\phi,\psi}(s_1, \cdots, s_l, t_1, \cdots, t_m) = \left\langle\!\!\left\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \phi, \, \psi \right\rangle\!\!\right\rangle . \tag{11}$$

Then $\eta_{\phi,\psi} \in E_{\mathbb{C}}^{\otimes(l+m)}$ and $(\phi, \psi) \mapsto \langle \kappa, \, \eta_{\phi,\psi} \rangle$ is a continuous bilinear form on (E) for any $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$. By general theory there exists a unique continuous linear operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\!\langle \Xi_{l,m}(\kappa)\phi, \, \psi \rangle\!\rangle = \langle \kappa, \, \eta_{\phi,\psi} \rangle , \qquad \phi, \psi \in (E). \tag{12}$$

In other words, $\Xi_{l,m}(\kappa)$ is defined through two canonical bilinear forms:

$$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\rangle = \left\langle \kappa, \left\langle\left\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\phi, \psi \right\rangle\right\rangle \right\rangle, \qquad \phi, \psi \in (E).$$

This suggests us to employ a formal integral expression:

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

We call $\Xi_{l,m}(\kappa)$ an *integral kernel operator* with *kernel distribution* κ. It is noteworthy that $\Xi_{l,m}(\kappa)$ is defined for any $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ and becomes a continuous operator in $\mathcal{L}((E), (E)^*)$. For any $p > 0$ with $|\kappa|_{-p} < \infty$ we have

$$\| \Xi_{l,m}(\kappa)\phi \|_{-p} \le C_{l,m;p} |\kappa|_{-p} \|\phi\|_p, \quad \phi \in (E), \tag{13}$$

where

$$C_{l,m;p} = \rho^{-p} \left(l^l m^m\right)^{1/2} \left(\frac{\rho^{-p}}{-2pe \log \rho}\right)^{(l+m)/2}.$$

This estimate is useful. Recall that $|\kappa|_{-p} < \infty$ for all sufficiently large $p > 0$.

The kernel distribution is not uniquely determined due to relation (10); however, for the uniqueness we only need to restrict ourselves to the subspace $(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ of all $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ which is symmetric with respect to the first l and the last m variables independently.

3 Symbol and Fock expansion

For $\Xi \in \mathcal{L}((E), (E)^*)$ a function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}, \tag{14}$$

is called the *symbol* of Ξ. Since the exponential vectors $\{\phi_\xi; \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of (E), the symbol recovers the operator uniquely. For an integral kernel operator,

$$\widehat{\Xi_{l,m}(\kappa)}(\xi, \eta) = \left\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle e^{\langle \xi, \eta \rangle}, \tag{15}$$

or equivalently,

$$\langle\langle \Xi_{l,m}(\kappa)\phi_\xi, \phi_\eta \rangle\rangle = \left\langle\left\langle \left\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle \phi_\xi, \phi_\eta \right\rangle\right\rangle, \tag{16}$$

where $\xi, \eta \in E_{\mathbb{C}}$ and $\kappa \in E_{\mathbb{C}}^{\otimes(l+m)}$ It is straightforward to see that $\Theta = \widehat{\Xi}$, $\Xi \in \mathcal{L}((E), (E)^*)$, possesses the following two properties:

(O1) For any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbb{C}}$, the function

$$z, w \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1), \qquad z, w \in \mathbb{C},$$

is entire holomorphic;

(O2) There exist constant numbers $C \ge 0$, $K \ge 0$ and $p \in \mathbb{R}$ such that

$$|\Theta(\xi, \eta)| \le C \exp K \left(|\xi|_p^2 + |\eta|_p^2\right), \qquad \xi, \eta \in E_{\mathbb{C}}.$$

More important is that the converse is also true.

Theorem 3.1 *Any $\mathbb{C}$-valued function Θ on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ satisfying conditions (O1) and (O2) is the symbol of an operator $\Xi \in \mathcal{L}((E),(E)^*)$, i.e., $\widehat{\Xi} = \Theta$.*

In fact, given such a function Θ, there exists a unique family of kernel distributions $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ such that

$$\Theta(\xi,\eta) = \sum_{l,m=0}^{\infty} \langle\langle \Xi_{l,m}(\kappa_{l,m})\phi_\xi,\, \phi_\eta \rangle\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Moreover, the series

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E), \tag{17}$$

converges in $(E)^*$, and thereby we obtain $\Xi \in \mathcal{L}((E),(E)^*)$ of which symbol is Θ. In particular, the symbol of $\Xi \in \mathcal{L}((E),(E)^*)$ satisfying (O1) and (O2), the above argument reproduces an operator Ξ in terms of integral kernel operators. Expression (17) is called the *Fock expansion* of Ξ.

In some practical problems operators on Fock space are only defined on the exponential vectors $\{\phi_\xi;\, \xi \in E_{\mathbb{C}}\}$ due to the fact that they are linearly independent. Theorem 3.1 is therefore crucial for checking whether the operator comes into our framework. In fact, our later discussion will depend on this point heavily. For detailed proof and further discussion see [10]. Here we do not mention anything about the case of $\mathcal{L}((E),(E))$ which is also important from some applications. For complete information see [11].

4 Operator-valued distributions

In [12] we studied $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$-valued distributions in general, where $\mathcal{E}$ is a standard CH-space. Here we recapitulate some results for $\mathcal{E} = (E)$.

Let $\{e_j\}_{j=0}^{\infty}$ be the normalized eigenfunctions of the operator A. For $\mathbf{i} = (i_1, \cdots, i_l)$ and $\mathbf{j} = (j_1, \cdots, j_m)$ we put

$$e(\mathbf{i}) = e_{i_1} \otimes \cdots \otimes e_{i_l}, \qquad e(\mathbf{j}) = e_{j_1} \otimes \cdots \otimes e_{j_m}.$$

For a linear map $L : E_{\mathbb{C}}^{\otimes(l+m)} \to \mathcal{L}((E),(E)^*)$ and $p,q,r,s \in \mathbb{R}$ we put

$$\| L \|_{l,m;p,q;r,s} = \sup \left\{ \sum_{\mathbf{i},\mathbf{j}} |\langle\langle L(e(\mathbf{i}) \otimes e(\mathbf{j}))\phi,\, \psi \rangle\rangle|^2\, |e(\mathbf{i})|_p^2\, |e(\mathbf{j})|_q^2\, ;\; \begin{matrix} \phi,\psi \in (E) \\ \|\phi\|_{-s} \le 1 \\ \|\psi\|_{-r} \le 1 \end{matrix} \right\}^{1/2}.$$

By definition for any $p,q,r,s \in \mathbb{R}$ we have

$$\sum_{\mathbf{i},\mathbf{j}} |\langle\langle L(e(\mathbf{i}) \otimes e(\mathbf{j}))\phi,\, \psi \rangle\rangle|^2\, |e(\mathbf{i})|_p^2\, |e(\mathbf{j})|_q^2 \le \| L \|_{l,m;p,q;r,s}^2\, \|\phi\|_{-s}^2\, \|\psi\|_{-r}^2 \tag{18}$$

and

$$\| L \|_{l,m;p,q;r,s} \le \rho^{lp'+mq'}\, \| L \|_{l,m;p+p',q+q';r+r',s+s'}\,, \qquad p',q',r',s' \ge 0. \tag{19}$$

For brevity we put

$$\| L \|_p = \| L \|_{l,m;p,p;p,p}, \qquad \| L \|_{l,m;p,q} = \| L \|_{l,m;p,q;p,q}.$$

The next result will be useful, for the proof see [12].

Proposition 4.1 *For a linear map $L : E_{\mathbb{C}}^{\otimes(l+m)} \to \mathcal{L}((E),(E)^*)$ the following four conditions are equivalent:*

(i) $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$;

(ii) $\sup\left\{ |\langle\langle L(\eta)\phi, \psi\rangle\rangle| ; \begin{array}{ll} \eta \in E_{\mathbb{C}}^{\otimes(l+m)}, & |\eta|_p \le 1 \\ \phi,\psi \in (E), & \|\phi\|_p \le 1, \|\psi\|_p \le 1 \end{array} \right\} < \infty$ *for some* $p \ge 0$;

(iii) $\| L \|_{-p} < \infty$ *for some* $p \ge 0$;

(iv) $\| L \|_{l,m;p,q;r,s} < \infty$ *for some* $p,q,r,s \in \mathbb{R}$.

In that case, for any $p,q,r,s \in \mathbb{R}$ we have

$$|\langle\langle L(\eta)\phi, \psi\rangle\rangle| \le \| L \|_{l,m;-p,-q;-r,-s} |\eta|_{l,m;p,q} \|\phi\|_s \|\psi\|_r, \tag{20}$$

and

$$\| L(\eta)\phi \|_{-r} \le \| L \|_{l,m;-p,-q;-r,-s} |\eta|_{l,m;p,q} \|\phi\|_s, \tag{21}$$

where $\eta \in E_{\mathbb{C}}^{\otimes(l+m)}$, $\phi,\psi \in (E)$.

Each $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$ is justifiably called an $\mathcal{L}((E),(E)^*)$-valued distribution on T^{l+m}. If $\mathcal{L}((E),(E)^*)$ were a Fréchet space, one would have a canonical isomorphism

$$\mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*)) \cong (E_{\mathbb{C}}^{\otimes(l+m)})^* \otimes \mathcal{L}((E),(E)^*)$$

by the kernel theorem; however, $\mathcal{L}((E),(E)^*)$ is not a Fréchet space. (It is known that $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$ is Fréchet if and only if $\mathcal{E}$ is a Hilbert space.)

5 Generalization of integral kernel operators

With each $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$ we associate an operator $\Xi \in \mathcal{L}((E),(E)^*)$ by the formula:

$$\langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m})\phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi,\eta \in E_{\mathbb{C}}. \tag{22}$$

We must check that the definition works; namely, conditions (O1) and (O2) in §3 are to be verified for

$$\Theta(\xi,\eta) = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m})\phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi,\eta \in E_{\mathbb{C}}. \tag{23}$$

In fact, the verification of (O1) is straightforward. As for (O2), it follows from (20) in Proposition 4.1 that

$$\begin{aligned} |\Theta(\xi,\eta)| &\le \| L \|_{-p} |\eta^{\otimes l} \otimes \xi^{\otimes m}|_p \|\phi_\xi\|_p \|\phi_\eta\|_p \\ &= \| L \|_{-p} |\eta|_p^l |\xi|_p^m \exp\frac{1}{2}\left(|\xi|_p^2 + |\eta|_p^2\right). \end{aligned}$$

Hence one may find $p \geq 0$, $K \geq 0$ and $C \geq 0$ such that

$$|\Theta(\xi,\eta)| \leq C \exp K\left(|\xi|_p^2 + |\eta|_p^2\right), \qquad \xi,\eta \in E_{\mathbb{C}},$$

which shows (O2). It then follows from the characterization theorem (Theorem 3.1) that Θ is the symbol of an operator $\Xi \in \mathcal{L}((E),(E)^*)$; namely, there exists a unique operator $\Xi \in \mathcal{L}((E),(E)^*)$ satisfying (22). It is reasonable to write

$$\Xi = \int_T \partial_{s_1}^* \cdots \partial_{s_l}^* L(s_1,\cdots,s_l,t_1,\cdots,t_m)\partial_{t_1}\cdots\partial_{t_m} ds_1\cdots ds_l dt_1\cdots dt_m. \tag{24}$$

The above constructed operator Ξ is a generalization of an integral kernel operator introduced in §2, compare (22) and (16).

This generalization occurs in an integral kernel operator. Let $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ and consider an integral kernel operator $\Xi_{l,m}(\kappa)$. To go further we need contraction of tensor products. For $g_l \in E_{\mathbb{C}}^{\otimes l}$ and $g_n \in E_{\mathbb{C}}^{\otimes n}$ we define $\kappa \otimes^l (g_l \otimes g_n) \in (E_{\mathbb{C}}^{\otimes(m+n)})^*$ as a unique element satisfying

$$\langle \kappa \otimes^l (g_l \otimes g_n), \zeta\rangle = \langle \kappa \otimes g_n, g_l \otimes \zeta\rangle, \qquad \zeta \in E_{\mathbb{C}}^{\otimes(m+n)}.$$

Then $\kappa \otimes^l g$ is defined for any $g \in E_{\mathbb{C}}^{\otimes(l+m)}$ by continuity and is called a left contraction. Moreover, it is easily verified that

$$|F \otimes^l g|_{-p} \leq \rho^{2pn} |F|_{-p} |g|_p, \qquad F \in (E_{\mathbb{C}}^{\otimes(l+m)})^*, \quad g \in E_{\mathbb{C}}^{\otimes(l+n)}. \tag{25}$$

The right contraction $\kappa \otimes_l g$ is similar. For detailed argument see [11].

Lemma 5.1 *Fix integers $0 \leq \alpha \leq l$ and $0 \leq \beta \leq m$. Given $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$,*

$$L_0(\eta_1,\cdots,\eta_\alpha,\xi_1,\cdots,\xi_\beta) = \Xi_{l-\alpha,m-\beta}\left((\kappa \otimes_\beta (\xi_1 \otimes\cdots\otimes \xi_\beta)) \otimes^\alpha (\eta_1 \otimes\cdots\otimes\eta_\alpha)\right)$$

becomes a continuous $(\alpha+\beta)$-linear map from $E_{\mathbb{C}}$ into $\mathcal{L}((E),(E)^*)$.

PROOF. For simplicity we put

$$\lambda = (\kappa \otimes_\beta (\xi_1 \otimes\cdots\otimes\xi_\beta)) \otimes^\alpha (\eta_1\otimes\cdots\otimes\eta_\alpha).$$

Take $p > 0$ with $|\kappa|_{-p} < \infty$. Then by (25) we have

$$|\lambda|_{-p} \leq |\kappa|_{-p} |\xi_1|_p \cdots |\xi_\beta|_p |\eta_1|_p \cdots |\eta_\alpha|_p. \tag{26}$$

On the other hand, in view of (13) we have

$$|\langle\langle L_0(\eta_1,\cdots,\eta_\alpha,\xi_1,\cdots,\xi_\beta)\phi, \psi\rangle\rangle| \leq C_{l-\alpha,m-\beta,p} |\lambda|_{-p} \|\phi\|_p \|\psi\|_p. \tag{27}$$

Given bounded subsets $B_1, B_2 \subset (E)$, we see from (26) and (27) that

$$\begin{aligned}&\sup_{\phi\in B_1,\psi\in B_2} |\langle\langle L_0(\eta_1,\cdots,\eta_\alpha,\xi_1,\cdots,\xi_\beta)\phi, \psi\rangle\rangle| \\ &\quad\leq C_{l-\alpha,m-\beta,p} \sup_{\phi\in B_1}\|\phi\|_p \sup_{\psi\in B_2}\|\psi\|_p |\kappa|_{-p} |\xi_1|_p\cdots|\xi_\beta|_p |\eta_1|_p\cdots|\eta_\alpha|_p,\end{aligned}$$

which implies the desired continuity of L_0. qed

By Lemma 5.1 there exists $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(\alpha+\beta)}, \mathcal{L}((E),(E)^*))$ such that

$$L(\eta_1 \otimes \cdots \otimes \eta_\alpha \otimes \xi_1 \otimes \cdots \otimes \xi_\beta) = L_0(\eta_1, \cdots, \eta_\alpha, \xi_1, \cdots, \xi_\beta).$$

In other words,

$$\begin{aligned} &L(\eta_1 \otimes \cdots \otimes \eta_\alpha \otimes \xi_1 \otimes \cdots \otimes \xi_\beta) \\ &\quad = \Xi_{l-\alpha,m-\beta}\left((\kappa \otimes_\beta (\xi_1 \otimes \cdots \otimes \xi_\beta)) \otimes^\alpha (\eta_1 \otimes \cdots \otimes \eta_\alpha)\right). \end{aligned} \tag{28}$$

Theorem 5.2 (FUBINI TYPE) *Fix integers $0 \le \alpha \le l$ and $0 \le \beta \le m$. Given $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ let $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(\alpha+\beta)}, \mathcal{L}((E),(E)^*))$ be defined as in (28). Then,*

$$\Xi_{l,m}(\kappa) = \int_{T^{\alpha+\beta}} \partial_{s_1}^* \cdots \partial_{s_\alpha}^* L(s_1, \cdots, s_\alpha, t_1, \cdots, t_\beta) \partial_{t_1} \cdots \partial_{t_\beta} ds_1 \cdots ds_\alpha dt_1 \cdots dt_\beta.$$

PROOF. The symbol of the right hand side is $\langle\langle L(\eta^{\otimes\alpha} \otimes \xi^{\otimes\beta})\phi_\xi, \phi_\eta \rangle\rangle$ by definition (22). In view of (28) we obtain

$$\begin{aligned} \left\langle\!\left\langle L(\eta^{\otimes\alpha} \otimes \xi^{\otimes\beta})\phi_\xi, \phi_\eta \right\rangle\!\right\rangle &= \left\langle\!\left\langle \Xi_{l-\alpha,m-\beta}((\kappa \otimes_\beta \xi^{\otimes\beta}) \otimes^\alpha \eta^{\otimes\alpha})\phi_\xi, \phi_\eta \right\rangle\!\right\rangle \\ &= \left\langle (\kappa \otimes_\beta \xi^{\otimes\beta}) \otimes^\alpha \eta^{\otimes\alpha}, \eta^{\otimes(l-\alpha)} \otimes \xi^{\otimes(m-\beta)} \right\rangle e^{\langle\xi,\eta\rangle} \\ &= \left\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle e^{\langle\xi,\eta\rangle}. \end{aligned}$$

The last expression coincides with the symbol of $\Xi_{l,m}(\kappa)$ by (15) and hence follows the assertion. qed

The above result is essential to discuss "canonical form" of an adapted operator-valued process. This topic will be discussed in a forthcoming paper.

6 Operator-valued Hitsuda-Skorokhod integrals

In this section we take

$$T = \mathbb{R}, \qquad A = 1 + t^2 - \frac{d^2}{dt^2}, \qquad E = \mathcal{S}(\mathbb{R}).$$

According to the discussion in the previous section we have a generalized integral kernel operator:

$$\int_T \partial_t^* L(t)\, dt, \qquad L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*)).$$

In this section we shall introduce a "stochastic integral" of the form:

$$\int_0^t \partial_s^* L(s)\, ds, \qquad t \ge 0.$$

For that purpose L should possess a stronger property that L is continuously extended to a linear map from $E_{\mathbb{C}}^*$ into $\mathcal{L}((E),(E)^*)$. Note the natural inclusion relation

$$\mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*)) \subset \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*)).$$

For $L \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$ we write $L_s = L(\delta_s)$ for simplicity. Then $\{L_s\}$ is regarded as an operator-valued (or quantum) stochastic process with values in $\mathcal{L}((E),(E)^*)$.

Lemma 6.1 *Let $L \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$. Then for any $f \in E_{\mathbb{C}}^*$ there exists an operator $M_f \in \mathcal{L}((E),(E)^*)$ such that*

$$\langle\langle M_f\phi_\xi,\, \phi_\eta\rangle\rangle = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Moreover, $f \mapsto M_f$ is continuous, i.e., $M \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$.

PROOF. First note that, for any $p \geq 0$ there exist $q > 0$ and $A_{p,q} \geq 0$ such that

$$|\,\xi\eta\,|_p \leq A_{p,q}\,|\,\xi\,|_{p+q}\,|\,\eta\,|_{p+q}\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Then, by duality we obtain

$$|\,f\eta\,|_{-(p+q)} \leq A_{p,q}\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q}\,, \qquad \eta \in E_{\mathbb{C}}, \quad f \in E_{\mathbb{C}}^*. \tag{29}$$

On the other hand, using the canonical isomorphism

$$\mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*)) \cong \mathcal{L}(E_{\mathbb{C}}^*, ((E)\otimes(E))^*),$$

which comes from the kernel theorem, we find $L^* \in \mathcal{L}((E)\otimes(E), E_{\mathbb{C}})$ such that

$$\langle\langle L(f)\phi,\, \psi\rangle\rangle = \langle f,\, L^*(\phi\otimes\psi)\rangle\,, \qquad f \in E_{\mathbb{C}}^*, \quad \phi,\psi \in (E).$$

By continuity, for any $p \geq 0$ there exist $q \geq 0$ and $B_{p,q} \geq 0$ such that

$$|\,L^*(\phi\otimes\psi)\,|_p \leq B_{p,q}\,\|\,\phi\,\|_{p+q}\,\|\,\psi\,\|_{p+q}\,, \qquad \phi,\psi \in (E). \tag{30}$$

We now consider

$$\Theta_f(\xi,\eta) = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle = \langle f\eta,\, L^*(\phi_\xi\otimes\phi_\eta)\rangle\,.$$

Suppose $p \geq 0$ is given arbitrarily. Take $q > 0$ with property (29). In view of (30) we may find $r \geq 0$ such that

$$\begin{aligned}
|\Theta_f(\xi,\eta)| &\leq |\,f\eta\,|_{-(p+q)}\,|\,L^*(\phi_\xi\otimes\phi_\eta)\,|_{p+q} \\
&\leq A_{p,q}\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q}\,B_{p+q,r}\,\|\,\phi_\xi\,\|_{p+q+r}\,\|\,\phi_\eta\,\|_{p+q+r} \\
&\leq A_{p,q}B_{p+q,r}\rho^r\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q+r}\exp\frac{1}{2}\left(|\,\xi\,|_{p+q+r}^2 + |\,\eta\,|_{p+q+r}^2\right).
\end{aligned}$$

Consequently, for any $p \geq 0$ we have found constants $C \geq 0$, $K \geq 0$ and $s \geq 0$ such that

$$|\Theta_f(\xi,\eta)| \leq C\,|\,f\,|_{-p}\exp K\left(|\,\xi\,|_{p+s}^2 + |\,\eta\,|_{p+s}^2\right), \qquad f \in E_{\mathbb{C}}^*, \quad \xi,\eta \in E_{\mathbb{C}}. \tag{31}$$

Hence by the characterization theorem (Theorem 3.1), for any $f \in E_{\mathbb{C}}^*$ there exists an operator $M_f \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\langle M_f\phi_\xi,\, \phi_\eta\rangle\rangle = \Theta_f(\xi,\eta) = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Obviously, $f \mapsto M_f$ is linear. Inequality (31) implies the continuity on the Hilbert space $\{f \in E_{\mathbb{C}}^*; |\,f\,|_{-p} < \infty\}$. Since $E_{\mathbb{C}}^*$ is the inductive limit of such Hilbert spaces, we conclude that $M \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$. qed

The operator M_f constructed above is denoted by

$$M_f = \int_T f(s)\partial_s^* L_s ds.$$

In particular, for $f = 1_{[0,t]}$ we write

$$\Omega_t \equiv \int_0^t \partial_s^* L_s ds, \qquad t \geq 0,$$

which forms a one-parameter family of operators in $\mathcal{L}((E),(E)^*)$. This is called an *operator-valued integral of Hitsuda-Skorokhod type.* To be sure we rephrase the definition:

$$\langle\langle \Omega_t \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle L(1_{[0,t]}\eta)\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{32}$$

It is interesting to observe how our operator-valued process $\{\Omega_t\}$ generalizes the Hitsuda-Skorokhod integral.

For that purpose we quote the definition of the Hitsuda-Skorokhod integral following [4]. Let $\Phi_t \in (E)^*$, $t \geq 0$, be given. Since $\partial_t^* \in \mathcal{L}((E)^*,(E)^*)$ for any t, for any $\phi \in (E)$ one obtains a function: $t \mapsto \langle\langle \partial_t^* \Phi_t, \phi \rangle\rangle$. Assume that the function is measurable and

$$\int_0^t |\langle\langle \partial_s^* \Phi_s, \phi \rangle\rangle| \, ds < \infty, \qquad t \geq 0.$$

Then, it is proved that there exists $\Psi_t \in (E)^*$, $t \geq 0$, uniquely such that

$$\langle\langle \Psi_t, \phi \rangle\rangle = \int_0^t \langle\langle \partial_s^* \Phi_s, \phi \rangle\rangle \, ds, \qquad \phi \in (E).$$

The above obtained Ψ_t is denoted by

$$\Psi_t = \int_0^t \partial_s^* \Phi_s \, ds$$

and is called the *Hitsuda-Skorokhod integral.* As is well known, the Hitsuda-Skorokhod integral coincides with the usual Itô integral when the integrand $\{\Phi_t\}$ is an adapted L^2-function with respect to the filtration generated by the Brownian motion

$$B_t(x) = \left\langle x, 1_{[0,t]} \right\rangle, \qquad x \in E^*, \quad t \geq 0.$$

In this connection see also [7], [8].

We need one more remark. Each $\Phi \in (E)^*$ gives rise to a continuous operator in $\mathcal{L}((E),(E)^*)$ by multiplication since $(\phi, \psi) \mapsto \phi\psi$ is a continuous bilinear map from $(E) \times (E)$ into (E). This identification extends to a natural inclusion relation $\mathcal{L}(E_{\mathbb{C}}^*, (E)^*) \subset \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$.

Now suppose we are given $\Phi \in \mathcal{L}(E_{\mathbb{C}}^*, (E)^*)$. Let $\widetilde{\Phi}$ denote the corresponding multiplication operator, i.e., $\widetilde{\Phi} \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$. Then one has an operator-valued integral of Hitsuda-Skorokhod type:

$$\Omega_t = \int_0^t \partial_s^* \widetilde{\Phi}_s ds, \qquad t \geq 0, \tag{33}$$

as well as the Hitsuda-Skorokhod integral in the original sense:

$$\Psi_t = \int_0^t \partial_s^* \Phi_s ds, \qquad t \geq 0. \tag{34}$$

In fact, since the both maps $t \mapsto \delta_t \in E^*$ and $t \mapsto \partial_t \phi \in (E)$ are continuous, so is $t \mapsto \langle\langle \partial_t^* \Phi_t, \phi \rangle\rangle$. Therefore Ψ_t is well defined.

Theorem 6.2 *For any $\Phi \in \mathcal{L}(E_{\mathbb{C}}^*, (E)^*)$ let Ω_t be the operator-valued integral of Hitsuda-Skorokhod type defined as in (33) and let Ψ be the Hitsuda-Skorokhod integral in the original sense defined as in (34). Then,*

$$\Psi_t = \Omega_t \phi_0, \qquad t \geq 0,$$

where ϕ_0 is the vacuum.

PROOF. By definition (32) we have

$$\langle\langle \Omega_t \phi_0, \phi_\eta \rangle\rangle = \left\langle\left\langle \tilde{\Phi}(1_{[0,t]}\eta)\phi_0, \phi_\eta \right\rangle\right\rangle = \left\langle\left\langle \Phi(1_{[0,t]}\eta), \phi_\eta \right\rangle\right\rangle.$$

In terms of the adjoint operator $\Phi^* \in \mathcal{L}((E), E_{\mathbb{C}})$ the last expression becomes

$$\left\langle\left\langle \Phi(1_{[0,t]}\eta), \phi_\eta \right\rangle\right\rangle = \left\langle 1_{[0,t]}\eta, \Phi^* \phi_\eta \right\rangle = \int_0^t \eta(s)(\Phi^*\phi_\eta)(s) ds.$$

Moreover, note that

$$\begin{aligned} \eta(s)(\Phi^*\phi_\eta)(s) &= \eta(s) \langle \delta_s, \Phi^*\phi_\eta \rangle = \eta(s) \langle\langle \Phi(\delta_s), \phi_\eta \rangle\rangle \\ &= \langle\langle \Phi(\delta_s), \partial_s \phi_\eta \rangle\rangle = \langle\langle \partial_s^* \Phi_s, \phi_\eta \rangle\rangle. \end{aligned}$$

Consequently,

$$\langle\langle \Omega_t \phi_0, \phi_\eta \rangle\rangle = \int_0^t \langle\langle \partial_s^* \Phi_s, \phi_\eta \rangle\rangle \, ds = \langle\langle \Psi_t, \phi_\eta \rangle\rangle,$$

and we come to $\Omega_t \phi_0 = \Psi_t$ as desired. qed

References

[1] N. N. Bogolubov, A. A. Logunov and I. T. Todorov: "Introduction to Axiomatic Quantum Field Theory," Benjamin, 1975.

[2] A. Grossman: *Fields at a point*, Commun. Math. Phys. **4** (1967), 203–216.

[3] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes, no. 13, Carleton University, Ottawa, 1975.

[4] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White Noise," Kluwer Academic, 1993.

[5] T. Hida, N. Obata and K. Saitô: *Infinite dimensional rotations and Laplacians in terms of white noise calculus*, Nagoya Math. J. **128** (1992), 65–93.

[6] R. L. Hudson and K. R. Parthasaraty: *Quantum Ito's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[7] I. Kubo: *Ito formula for generalized Brownian functionals*, in "Theory and Application of Random Fields (G. Kallianpur ed.)," pp. 156–166, Lect. Notes in Math. Vol. 49, Springer-Verlag, 1983.

[8] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I–IV*, Proc. Japan Acad. **56A** (1980), 376–380; 411-416; **57A** (1981), 433–437; **58A** (1982), 186–189.

[9] P. A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer-Verlag, 1993.

[10] N. Obata: *An analytic characterization of symbols of operators on white noise functionals*, J. Math. Soc. Japan **45** (1993), 421–445.

[11] N. Obata: "Elements of White Noise Calculus," Lect. Notes in Math., Springer-Verlag, to be published.

[12] N. Obata: *Operator calculus on vector-valued white noise functionals*, to appear in J. Funct. Anal. **120** (1994).

[13] K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1993.

[14] K. R. Parthasarathy and K. B. Sinha: *Stochastic integral representation of bounded quantum martingales in Fock space*, J. Funct. Anal. **67** (1986), 126–151.