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集合値写像の錐凸性について*

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Abstract: In this paper, we define various kinds of cone convexity for set-valued maps as generalizations of classical concepts of cone convexity for vector-valued maps; that is, convexity, convexlikeness, quasiconvexity, properly quasiconvexity, and naturally quasiconvexity. Moreover, we investigate some relations among those kinds of cone convexity for set-valued maps. Especially, we show that, among some kinds of cone convexity for set-valued maps, there are similar relations to those among the corresponding cone convexities for vector-valued maps.

Key words: Set-valued analysis, convexity of set-valued maps.

1. INTRODUCTION

How is the concept of convexity of set-valued map defined? In this paper, we propose some methods and useful symbols to define concepts of convexity of set-valued maps. If f is a vector-valued map, concepts of convexity are based on vector-ordering for two vector. On the other hand, the case of set-valued map is not so simple, because we should compare two image sets with respect to vector-ordering. For set-valued maps, we know some generalized concepts of convex of vector-valued maps are proposed to extend optimal conditions in the area of optimization theory; [2, 3, 4, 5, 9, 11, 12, 16, 17]. Such generalizations are natural and useful for optimization problems, but there is no detail report about unified theory for convexity of set-valued maps. Therefore, the aim of this paper is to give a unified report on such convexity, that is, we define five kinds of cone convexity for set-valued maps as generalizations of some convexities for vector-valued maps, and we investigate relationship among such cone convexities.

The organization of the paper is as follows. In Section 2, we consider some concepts of comparison of two sets with respect to a vector ordering, an we introduce six kinds of relations. In Section 3, based on each of the six relationships, we introduce five categories of cone convexity for set-valued maps as generalizations of some convexities for vector-valued maps; convexity, convexlikeness, quasiconvexity, properly quasiconvexity and naturally quasiconvexity for set-valued maps. It is simple to define convexities, convexlikenesses and quasiconvexities of set-valued maps, however, the concepts of the others, that is, properly quasiconvexities and naturally quasiconvexities for set-valued maps are more complicated. Because convexity, convexlikeness, and quasiconvexity for vector-valued maps are represented by conditions between

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two vectors, however, properly quasiconvexity and naturally quasiconvexity are defined by conditions between a vector and a subset. Moreover, we investigate some relations among those kinds of cone convexity for set-valued maps. Especially, we show that, among some kinds of cone convexity for set-valued maps, there are similar relations to those among the corresponding cone convexities for vector-valued maps.

2. RELATIONSHIP BETWEEN TWO SETS WITH RESPECT TO CONES

Throughout this paper, let Z be an ordered topological vector space with the vector ordering $\leq_C$ induced by a convex cone C: for $x, y \in Z$,

$$x \leq_C y \text{ if } y - x \in C. \tag{2.1}$$

The convex cone C is assumed not to be pointed but to be solid, that is, its topological interior int C is nonempty; hence, $C^0 := (\text{int } C) \cup \{0\}$ is a pointed convex cone and induces another antisymmetric vector ordering $\leq_{C^0}$ weaker than $\leq_C$ in Z. Also, F is said to be a set-valued map from X into Z if F is a map from X into 2^Z, which is the power set of Z, and also we write $F : X \rightsquigarrow Z$. Moreover, for a set-valued map $F : X \rightsquigarrow Z$ we use the following symbols:

$$\text{Graph}(F) := \{(x, y) \mid x \in X, y \in F(x)\}; \quad \text{Dom}F := \{x \in X \mid F(x) \neq \emptyset\}. \tag{2.2}$$

In this paper, we consider several generalizations of convexity of vector-valued function into that of set-valued map. With respect to convexity of function there are two ways of generalization. One is a generalization based on values of set-valued map F, that is, a prescription of relationship between two sets $F(\lambda x_1 + (1-\lambda)x_2)$ and $\lambda F(x_1) + (1-\lambda)F(x_2)$; the other is a generalization based on equivalent characteristic sets of set-valued map F, that is, prescriptions by epigraph of F, image set of F, and lower level set of F. This paper's approach is the former, because the latter generalization is included in the former as mentioned in Section 3.

Now, we start with discussion on set-relationship, that is, we introduce eight kinds of relationships between two sets in an ordered vector space with respect to a convex cone. This classification is based on two ideas for set-relation.

First, with respect to relationship between two vectors $a, b \in Z$, one of the followings holds:

(i) $a \in b + C$ (equivalently $b \in a - C$); (iii) $b \in a + C$ (equivalently $a \in b - C$);

(ii) $a \notin b + C$ (equivalently $b \notin a - C$); (iv) $b \notin a + C$ (equivalently $a \notin b - C$).

These relationships are summarized as $b \leq_C a$, $b \not\leq_C a$ or $a \leq_C b$, $a \not\leq_C b$, that is, one vector is dominated by the other vector or otherwise. In the case of relationship between a nonempty set $A \subset Z$ and a vector $b \in Z$, a different situation is observed; we have two domination structure

(i) for all $a \in A$, $a \leq_C b$;

(ii) there exists $a \in A$ such that $a \leq_C b$.

The first relation means the vector b dominates the whole set A from above with respect to the vector ordering $\leq_C$. The second relation means the vector b is dominated from below by an element of the set A. If the set A is singleton, they are coincident with each other. These relationships are denoted by $b \in A \overset{+}{\cap} C$ and $b \in A \uplus C$, respectively, where

$$A \overset{+}{\cap} C := \bigcap_{a \in A} (a + C) \quad \text{and} \quad A \uplus C := \bigcup_{a \in A} (a + C). \tag{2.3}$$

Analogously, we use the following notations for a nonempty set $B \subset Z$:

$$B \overset{-}{\cap} C := \bigcap_{b \in B} (b - C) = B \overset{+}{\cap} (-C) \quad \text{and} \quad B \overset{-}{\cup} C := \bigcup_{b \in B} (b - C) = B \uplus (-C). \tag{2.4}$$

It is easy to see that $A \overset{+}{\cap} C \subset A \uplus C$ and $B \overset{-}{\cap} C \subset B \overset{-}{\cup} C$, and also that $A \uplus B = A + B$ and $A \overset{-}{\cup} B = A - B$.

Secondly, we consider the relationship between two nonempty sets in Z, which is strongly concerned with intersection and inclusion in set theory. Given nonempty sets $A, B \subset Z$, exactly one of following conditions holds: (i) $A \cap B = \emptyset$; (ii) $A \cap B \neq \emptyset$. The latter case includes its special cases $A \subset B$ and $A \supset B$.

By using above two ideas, we classify the relationship between two nonempty sets $A, B \in Z$ in the sense that A is (partially) dominated from above by B or A (partially) dominates B from below:

(i) $A \subset B \overset{-}{\cap} C$;

(ii) $A \cap (B \overset{-}{\cap} C) \neq \emptyset$;

(iii) $A \uplus C \supset B$;

(iv) $(A \uplus C) \cap B \neq \emptyset$;

(v) $A \overset{+}{\cap} C \supset B$;

(vi) $(A \overset{+}{\cap} C) \cap B \neq \emptyset$;

(vii) $A \subset B \overset{-}{\cup} C$;

(viii) $A \cap (B \overset{-}{\cup} C) \neq \emptyset$.

Since conditions (i) and (v) coincide and conditions (iv) and (viii) coincide, we define six kinds of classification for set-relationship; see Figure 1.

Definition 2.1. For nonempty subsets A, B of Z, we denote

- $A \overset{+}{\cap} C \supset B$ by $A \leq_C^{(\mathrm{i})} B$;
- $A \cap (B \overset{-}{\cap} C) \neq \emptyset$ by $A \leq_C^{(\mathrm{ii})} B$;
- $A \uplus C \supset B$ by $A \leq_C^{(\mathrm{iii})} B$;
- $(A \overset{+}{\cap} C) \cap B \neq \emptyset$ by $A \leq_C^{(\mathrm{iv})} B$;
- $A \subset B \overset{-}{\cup} C$ by $A \leq_C^{(\mathrm{v})} B$;
- $(A \uplus C) \cap B \neq \emptyset$ by $A \leq_C^{(\mathrm{vi})} B$.

As shown in Figure 1, all implications among the set-relations are easily verified.

Proposition 2.1. For nonempty subsets A, B, the following statements hold:

- $A \leq_C^{(\mathrm{i})} B$ implies $A \leq_C^{(\mathrm{ii})} B$;
- $A \leq_C^{(\mathrm{ii})} B$ implies $A \leq_C^{(\mathrm{iii})} B$;
- $A \leq_C^{(\mathrm{iii})} B$ implies $A \leq_C^{(\mathrm{vi})} B$;
- $A \leq_C^{(\mathrm{i})} B$ implies $A \leq_C^{(\mathrm{iv})} B$;
- $A \leq_C^{(\mathrm{iv})} B$ implies $A \leq_C^{(\mathrm{v})} B$;
- $A \leq_C^{(\mathrm{v})} B$ implies $A \leq_C^{(\mathrm{vi})} B$.

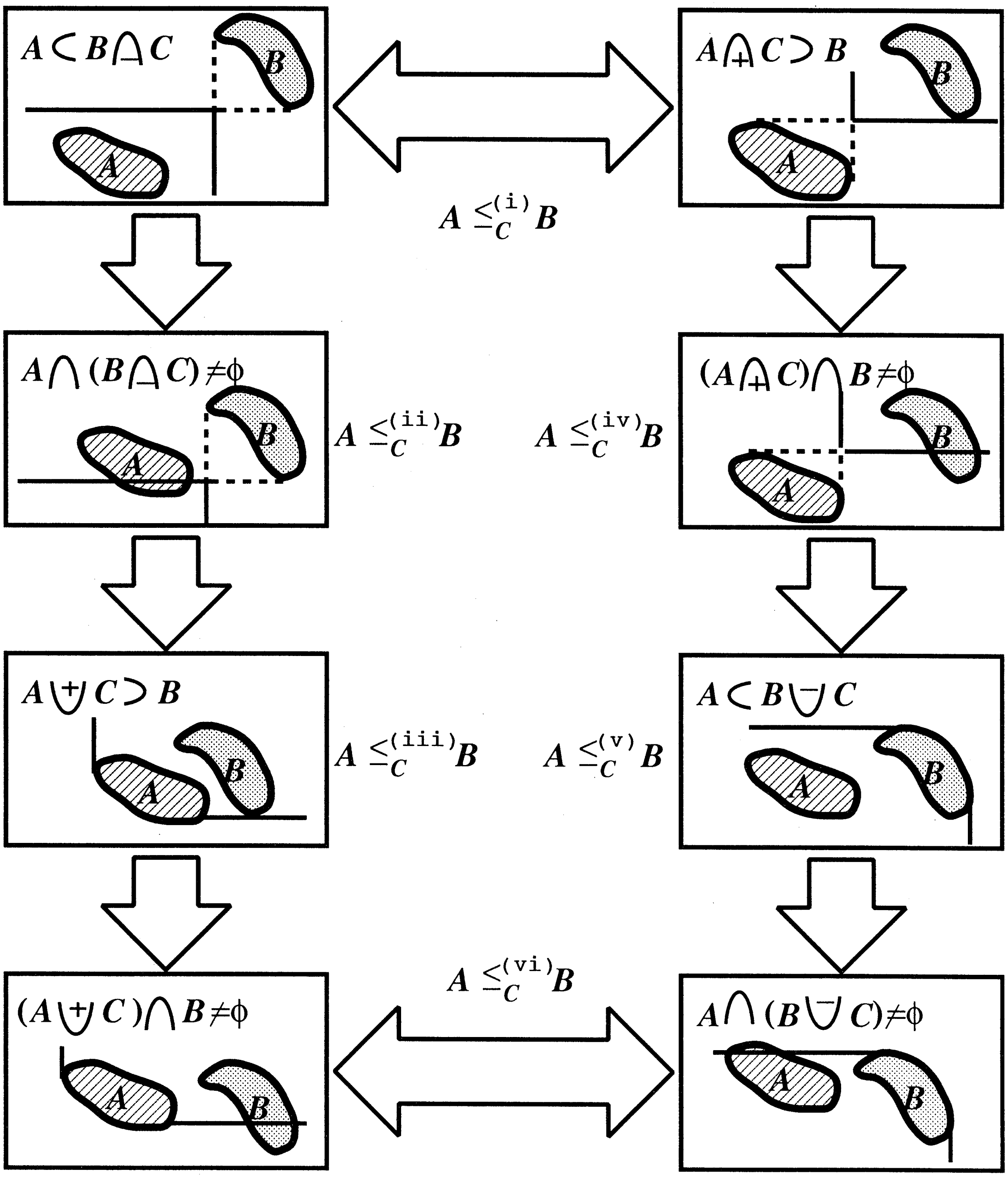

Figure 1: **Six kinds of classification for set-relationship**

3. CATEGORIZED CONVEXITY FOR SET-VALUED MAPS

In this paper, convexity of set-valued maps is generalized in the following two ways: One is based on prescriptions of relationship between two sets $F(\lambda x_1 + (1-\lambda)x_2)$ and $\lambda F(x_1) + (1-\lambda)F(x_2)$; the other is based on prescriptions by epigraph of F, image set of F, and lower level set of F. Epigraph convexity, Image-set convexity, and lower level-set convexity are concerned with convexity, convexlikeness, and quasiconvexity of set-valued map, respectively.

Using the six kinds of relationships between two nonempty sets introduced in Section 2, we consider some different concepts with respect to six different set-relations $\leq_C^{(k)}$ ($k = $ i, ..., vi) for each convexity of set-valued map as generalizations of those of vector-valued function. We categorize such generalized convexities into five class, that is, convexity, convexlikeness, quasiconvexity, properly quasiconvexity, naturally quasiconvexity; and this section consists of four subsections related to them.

3.1. CONVEXITY AND CONVEXLIKENESS OF SET-VALUED MAP

A vector-valued function $f : X \to Z$ is said to be C-convex ([14, 15]) if for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda)f(x_2), \tag{3.1}$$

which is equivalent to the following condition:

$$\mathrm{Graph}(f) + \{\theta_X\} \times C \ \text{ is a convex set.} \tag{3.2}$$

Whenever $Z = \boldsymbol{R}$ and $C = \boldsymbol{R}_+$, C-convexity above is the same as the ordinary convexity of a real-valued function. Based on the six different set-relations $\leq_C^{(k)}$ ($k = $ i, ..., vi), we propose the following generalization of convexity (3.1) to set-valued map.

DEFINITION 3.1. A set-valued map $F : X \rightsquigarrow Z$ is said to be

- **type (k) convex** ($k = $ i, ..., vi) if for every $x_1, x_2 \in \mathrm{Dom}F$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} \lambda F(x_1) + (1-\lambda)F(x_2); \tag{3.3}$$

- **graphical-convex** if $\mathrm{Graph}(F) + (\{\theta_X\} \times C)$ is a convex set. (3.4)

We have some implications among convexities above:

PROPOSITION 3.1. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccccc}
\text{type (i) convex} & & \longrightarrow & & \text{type (iv) convex} \\
\downarrow & & & & \downarrow \\
\text{type (ii) convex} & & & & \text{type (v) convex} \\
\downarrow & & & & \downarrow \\
\text{type (iii) convex} & \longleftrightarrow & \text{graphical-convex} & \longrightarrow & \text{type (vi) convex}
\end{array}$$

Next, we proceed to the convexlikeness of set-valued map. A vector-valued function $f : X \to Z$ is said to be C-convexlike ([14, 15]) if for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $x \in X$ such that

$$f(x) \leq_C \lambda f(x_1) + (1-\lambda) f(x_2), \tag{3.5}$$

which is equivalent to the following condition:

$$f(X) + C \ \text{ is a convex set.} \tag{3.6}$$

Based on the six different set-relations $\leq_C^{(k)}$ $(k = \text{i}, \ldots, \text{vi})$, we propose the following generalization of convexlikeness (3.5) to set-valued map.

DEFINITION 3.2. A set-valued map $F : X \rightsquigarrow Z$ is said to be

- **type (k) convexlike** $(k = \text{i}, \ldots, \text{vi})$ if for every $x_1, x_2 \in \mathrm{Dom}F$ and $\lambda \in (0,1)$, there exists $x \in \mathrm{Dom}F$ such that

$$F(x) \leq_C^{(k)} \lambda F(x_1) + (1-\lambda) F(x_2); \tag{3.7}$$

- **graphical-convexlike** if $F(\mathrm{Dom}(F)) + C$ is a convex set. (3.8)

We have some implications among convexlikeness above:

PROPOSITION 3.2. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccccc}
\text{type (i) convexlike} & & \longrightarrow & & \text{type (iv) convexlike} \\
\downarrow & & & & \downarrow \\
\text{type (ii) convexlike} & & & & \text{type (v) convexlike} \\
\downarrow & & & & \downarrow \\
\text{type (iii) convexlike} & \longrightarrow & \text{graphical-convexlike} & \longrightarrow & \text{type (vi) convexlike}
\end{array}$$

PROOF. By Proposition 2.1, we can show that the above relations among type (k) convexlikenesses. Next, we show type (iii) convexlikeness implies graphical-convexlikeness and graphical-convexlikeness implies type (vi) convexlikeness. We can see that F is graphical-convexlike if and only if for each x_1, $x_2 \in \mathrm{Dom}(F)$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ and $\lambda \in (0,1)$, there exist $x \in \mathrm{Dom}(F)$ and $y \in F(x)$ such that $y \leq_C \lambda y_1 + (1-\lambda) y_2$. From this and definitions of type (k) convexlikeness, the claim is proved. □

PROPOSITION 3.3. For a set-valued map $F : X \rightsquigarrow Z$ and each $k = \text{i}, \ldots, \text{vi}$, type (k) convexity implies type (k) convexlikeness.

3.2. QUASI CONVEXITY OF SET-VALUED MAP

A vector-valued function $f : X \to Z$ is said to be quasi C-convex ([14, 15]) if it satisfies one of the following two equivalent conditions:

- **(Luc's quasi C-convexity)** for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$f(\lambda x_1 + (1-\lambda) x_2) \leq_C z, \qquad \text{for all } z \in C(f(x_1), f(x_2)), \tag{3.9}$$

where $C(f(x_1), f(x_2)) = \{z \in Z \mid f(x_1) \leq_C z, \quad f(x_2) \leq_C z\}$;

- **(Ferro's quasi C-convexity)** for each $z \in Z$, the set

$$f^{-1} = \{x \in X \mid f(x) \in z - C\} \text{ is convex.} \tag{3.10}$$

Based on the six different set-relations $\leq_C^{(k)}$ ($k =$ i, ..., vi), we propose two ways of generalization of quasi C-convexities (3.9) and (3.10) to set-valued map.

First, to define Luc's type quasiconvexity of set-valued map we introduce the following sets. For a set-valued map $F : X \to Z$ and $x_1, x_2 \in \mathrm{Dom}F$, we denote, respectively, the dominated set from below by sets $F(x_1)$ and $F(x_2)$ and the set of points dominating sets $F(x_1)$ and $F(x_2)$ simultaneously from above by

$$C_L(F(x_1), F(x_2)) = (F(x_1) \uplus C) \cap (F(x_2) \uplus C), \tag{3.11}$$

and

$$C_U(F(x_1), F(x_2)) = (F(x_1) \Cap C) \cap (F(x_2) \Cap C). \tag{3.12}$$

When F is a single-valued map, we can verify that

$$C_L(F(x_1), F(x_2)) = C_U(F(x_1), F(x_2)) = C(F(x_1), F(x_2)). \tag{3.13}$$

By using two sets and the six different set-relations $\leq_C^{(k)}$ ($k =$ i, ..., vi), we consider generalization of quasi C-convexity (3.9), but types (iv)–(vi) generalizations are meaningless since the following conditions (3.14) and (3.15) are trivial in the cases.

DEFINITION 3.3. For each $k =$ i, ii, iii, a set-valued map $F : X \rightsquigarrow Z$ is said to be

- **type (k)-lower quasiconvex** if for every $x_1, x_2 \in \mathrm{Dom}F$ and $\lambda \in (0, 1)$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(k)} C_L(F(x_1), F(x_2)); \tag{3.14}$$

- **type (k)-upper quasiconvex** if for every $x_1, x_2 \in \mathrm{Dom}F$ and $\lambda \in (0, 1)$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_C^{(k)} C_U(F(x_1), F(x_2)). \tag{3.15}$$

Second, we define Ferro's type quasiconvexity of set-valued map.

DEFINITION 3.4. A set-valued map $F : X \rightsquigarrow Z$ is said to be

- **Ferro type (-1)-quasiconvex** if for every $z \in Z$,

$$F^{-1}(z - C) := \{x \in X \mid F(x) \cap (z - C) \neq \emptyset\} \text{ is convex;} \tag{3.16}$$

- **Ferro type ($+1$)-quasiconvex** if for every $z \in Z$,

$$F^{+1}(z - C) := \{x \in X \mid F(x) \subset (z - C)\} \text{ is convex.} \tag{3.17}$$

These sets are said to be the lower level sets of set-valued map F, and Ferro type (-1)-quasiconvexity and Ferro type $(+1)$-quasiconvexity are provided by convexity of their sets, respectively. By Proposition 2.1. and simple demonstration, we have the following interesting implications among quasiconvexities above, including the level-set convexity.

PROPOSITION 3.4. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccccccc} & & \text{type (i)-lower quasiconvex} & \to & \text{type (i)-upper quasiconvex} & \leftrightarrow & \text{Ferro type } (+1)\text{-quasiconvex} \\ & & \downarrow & & \downarrow & & \\ & & \text{type (ii)-lower quasiconvex} & \to & \text{type (ii)-upper quasiconvex} & & \\ & & \downarrow & & \downarrow & & \\ \text{Ferro type } (-1)\text{-quasiconvex} & \leftrightarrow & \text{type (iii)-lower quasiconvex} & \to & \text{type (iii)-upper quasiconvex} & & \end{array}$$

3.3. PROPERLY QUASI CONVEXITY OF SET-VALUED MAP

A vector-valued function $f : X \to Z$ is said to be properly quasi C-convex ([14, 15]) if for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C f(x_1) \text{ or } f(\lambda x_1 + (1-\lambda)x_2) \leq_C f(x_2). \tag{3.18}$$

This condition can be described in another way, $f(\lambda x_1 + (1-\lambda)x_2) \in (\{f(x_1), f(x_2)\} - C)$, and hence various types of generalization of the properly quasiconvexity can be considered, but we concentrate upon a generalization of properly quasi C-convexity (3.18) to set-valued map.

DEFINITION 3.5. For each $k = \text{i}, \ldots, \text{vi}$, a set-valued map $F : X \rightsquigarrow Z$ is said to be **type (k) properly quasiconvex** if for every $x_1, x_2 \in \text{Dom}F$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} F(x_1) \text{ or } F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} F(x_2). \tag{3.19}$$

By Proposition 2.1, we have some implications among properly quasiconvexities above:

PROPOSITION 3.5. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccc} \text{type (i) properly quasiconvex} & \longrightarrow & \text{type (iv) properly quasiconvex} \\ \downarrow & & \downarrow \\ \text{type (ii) properly quasiconvex} & & \text{type (v) properly quasiconvex} \\ \downarrow & & \downarrow \\ \text{type (iii) properly quasiconvex} & \longrightarrow & \text{type (vi) properly quasiconvex} \end{array}$$

3.4. NATURALLY QUASI CONVEXITY OF SET-VALUED MAP

A vector-valued function $f : X \to Z$ is said to be naturally quasi C-convex ([14, 15]) if for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C \mu f(x_1) + (1-\mu) f(x_2). \tag{3.20}$$

This condition can be described in another way, $f(\lambda x_1 + (1-\lambda)x_2) \in (\mathrm{co}\,\{f(x_1), f(x_2)\} - C)$, and hence various types of generalization of the naturally quasiconvexity can be considered, but we concentrate upon a generalization of naturally quasi C-convexity (3.20) to set-valued map.

DEFINITION 3.6. For each $k =$ i, ..., vi, a set-valued map $F : X \rightsquigarrow Z$ is said to be **type (k) naturally quasiconvex** if for every $x_1, x_2 \in \mathrm{Dom}F$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} \mu F(x_1) + (1-\mu)F(x_2). \tag{3.21}$$

By Proposition 2.1, we have some implications among naturally quasiconvexities above:

PROPOSITION 3.6. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccc}
\text{type (i) naturally quasiconvex} & \longrightarrow & \text{type (iv) naturally quasiconvex} \\
\downarrow & & \downarrow \\
\text{type (ii) naturally quasiconvex} & & \text{type (v) naturally quasiconvex} \\
\downarrow & & \downarrow \\
\text{type (iii) naturally quasiconvex} & \longrightarrow & \text{type (vi) naturally quasiconvex}
\end{array}$$

Finally, we have the following results on the relationships among the generalized convexities of set-valued map introduced in the paper, see [8].

THEOREM 3.1. For a set-valued map $F : X \rightsquigarrow Z$, the following statements hold:

(i) For each $k =$ i, ..., vi, type (k) convexity implies type (k) convexlikeness;

(ii) For each $k =$ i, ..., vi, type (k) convexity implies type (k) naturally quasiconvexity;

(iii) For each $k =$ i, ..., vi, type (k) properly quasiconvexity implies type (k) naturally quasiconvexity;

(iv) Type (iii) naturally quasiconvexity implies type (iii)-lower quasiconvexity;

(v) Type (vi) naturally quasiconvexity implies type (ii)-upper quasiconvexity;

(vi) Assume that C is a closed convex cone and that F is an upper semicontinuous and convex-valued set-valued map. If F is type (iii) naturally quasiconvex then it is also type (iii) convexlike.

These results are similar to those of vector-valued versions; see [14, 15].

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