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Laplace tranform and Fourier-Sato tranform

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Review on formal and moderate cohomology. Let M be a real manifold, and let R-$Cons(M)$ denote the category of R-constructible sheaves on M, $D^b_{\mathrm{R}-c}(\mathrm{C}_M)$ its derived category. Recall first the functors $\mathcal{T}hom(\cdot, \mathcal{D}b_M)$ of [4] and the dual functor $\cdot \overset{w}{\otimes} \mathcal{C}^\infty_M$ of [6], defined on the category R-$Cons(M)$, with values in the category $Mod(\mathcal{D}_M)$ of $\mathcal{D}_M$-modules on M. (The first functor is contravariant).

They are characterized as follows. Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M and by $\mathcal{C}^\infty_M$ the sheaf of $\mathcal{C}^\infty$ functions on M. Let Z (resp. U) be a closed (resp. open) subanalytic subset of M. Then these two functors are exact and moreover:

$$\begin{aligned} \mathcal{T}hom(\mathrm{C}_Z, \mathcal{D}b_M) &= \Gamma_Z(\mathcal{D}b_M), \\ \mathrm{C}_U \overset{w}{\otimes} \mathcal{C}^\infty_M &= \mathcal{I}^\infty_{M\setminus U}, \end{aligned}$$

where $\Gamma_Z(\mathcal{D}b_M)$ denotes as usual the subsheaf of $\mathcal{D}b_M$ of sections supported by Z and $\mathcal{I}^\infty_{M\setminus U}$ denotes the ideal of $\mathcal{C}^\infty_M$ of sections vanishing up to order infinity on $M\setminus U$.

These functors being exact, they extend naturally to the derived category $D^b_{\mathrm{R}-c}(\mathrm{C}_X)$. We keep the same notations to denote the derived functors.

Now let X be a complex manifold and denote by $\overline{X}$ the complex conjugate manifold and by X_{R} the real underying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on X. The functors of moderate and formal cohomology (see [4], [6]) are defined for $F \in D^b_{\mathrm{R}-c}(\mathrm{C}_{X_{\mathrm{R}}})$ by:

$$\begin{aligned} \mathcal{T}hom(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_{X_{\mathrm{R}}})) \\ F \overset{w}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{w}{\otimes} \mathcal{C}^\infty_{X_{\mathrm{R}}}). \end{aligned}$$

Laplace transform. Consider a complex vector space E of complex dimension n, and denote by $j : \mathrm{E} \hookrightarrow \mathrm{P}$ its projective compactification. Let $D^b_{\mathrm{R}-c,\mathrm{R}^+}(\mathrm{C}_{\mathrm{E}})$ denote the full triangulated subcategory of $D^b_{\mathrm{R}-c}(\mathrm{C}_{\mathrm{E}})$ consisting of R^+-conic objects (i.e.

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objects whose cohomology is R-constructible and locally constant on the orbits of the action of R^+ on E).

Let $F \in D^b_{\mathrm{R}-c,\mathrm{R}^+}(\mathrm{C_E})$ and set for short

$$\mathrm{THom}\,(F, \mathcal{O}_\mathrm{E}) = \mathrm{R}\Gamma(\mathrm{P}; \mathcal{T}hom(j_!F, \mathcal{O}_\mathrm{P}))$$
$$\mathrm{WTens}(F, \mathcal{O}_\mathrm{E}) = \mathrm{R}\Gamma(\mathrm{P}; j_!F \overset{w}{\otimes} \mathcal{O}_\mathrm{P})$$

These are objects of the bounded derived category $D^b(W(\mathrm{E}))$ of the category of modules over the Weyl algebra $W(\mathrm{E})$. Let E^* denote the dual vector space to E. One denotes by $F^\wedge$ the Fourier-Sato transform of the sheaf F,an object of $D^b_{\mathrm{R}-c,\mathrm{R}^+}(\mathrm{C}_{\mathrm{E}^*})$. (see [5] for an exposition). One identifies $D^b(W(\mathrm{E}^*))$ to $D^b(W(\mathrm{E}))$ by the usual Fourier transform.

Theorem 0.1. *The classical Laplace transform extends naturally as isomorphisms in $D^b(W(\mathrm{E}))$:*

$$L_\mathrm{E} : \mathrm{THom}\,(F, \mathcal{O}_\mathrm{E}) \simeq \mathrm{THom}\,(F^\wedge[n], \mathcal{O}_{\mathrm{E}^*}) \tag{0.1}$$
$$L_\mathrm{E} : \mathrm{WTens}(F, \mathcal{O}_\mathrm{E}) \simeq \mathrm{WTens}(F^\wedge[n], \mathcal{O}_{\mathrm{E}^*}). \tag{0.2}$$

Applications 1. Let M be a real vector space of dimension n such that E is a complexification of M. As a particular case of the theorem, we obtain a characterization of the Laplace transform of the space of distributions on M supported by (not necessarily convex) cones. Let γ denote a closed subanalytic cone in M and let $\Gamma_\gamma \mathcal{S}'_M$ denote the space of tempered distributions supported by γ. One has $\Gamma_\gamma \mathcal{S}'_M \simeq \mathrm{THom}\,(\mathrm{C}_\gamma[-n], \mathcal{O}_\mathrm{E})$. Hence, we get that the Laplace transform of distributions induces an isomorphism:

$$L_\mathrm{E} : \Gamma_\gamma \mathcal{S}'_M \simeq \mathrm{THom}\,((\mathrm{C}_\gamma)^\wedge, \mathcal{O}_{\mathrm{E}^*}) = .$$

When γ is proper and convex, this result is well known, since $(\mathrm{C}\gamma)^\wedge \simeq \mathrm{C}_U$ where U is the open convex tube $int\gamma^0 \times \sqrt{-1}M^*$, the interior of the polar cone to γ, and the right hand side denotes the space of holomorphic functions in this tube, tempered up to infinity. When $\gamma = M$, one recovers the isomorphism between $\mathcal{S}'_M$ and $\mathcal{S}'_{\sqrt{-1}M^*}$.

Let us consider now the case where γ is a non degenerate quadratic cone. Let (x', x'') denote the coordinates on $M = \mathrm{R}^n = \mathrm{R}^p \times \mathrm{R}^q$ with $p, q \geq 1$, and let $\gamma = \{x; x'^2 - x''^2 \leq 0\}$. Let (u', u'') denote the dual coordinates on M^*, and let $\lambda = \{(u', u''); u'^2 - u''^2 \geq 0\}$. One checks easily that $(\mathrm{C}_\gamma)^\wedge \simeq \mathrm{C}_\lambda[-q]$. We get the isomorphi= sm:

$$L_\mathrm{E} : \Gamma_\gamma \mathcal{S}'_M \simeq H^q\,\mathrm{THom}\,(\mathrm{C}_{\lambda \times \sqrt{-1}M^*}, \mathcal{O}_{\mathrm{E}^*}).$$

This last result is essentially due to Faraut-Gindikin [2] (in a different langage and with a different proof).

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Applications 2. Denote by $\mathcal{O}^t_{\mathrm{E}}$ and $\mathcal{O}^w_{\mathrm{E}}$ the conic sheaves on E associated to the presheaves $U \mapsto \mathrm{THom}\,(\mathrm{C}_U, \mathcal{O}_{\mathrm{E}})$ and $U \mapsto \mathrm{WTens}(\mathrm{C}_{\bar{U}}, \mathcal{O}_{\mathrm{E}})$, respectively. One easily deduces from the main theorem that the Laplace transform induces isomorphism= s:

$$\begin{aligned}(\mathcal{O}^t_{\mathrm{E}})^\wedge[n] &\simeq \mathcal{O}^t_{\mathrm{E}^*},\\ (\mathcal{O}^w_{\mathrm{E}})^\wedge[n] &\simeq \mathcal{O}^w_{\mathrm{E}^*}.\end{aligned}$$

This gives a new proof of a result of Hotta-Kashiwara [3] and Brylinski-Malgrange-Verdier [1] on the Fourier-Sato transform of the sheaf of holomorphic solutions of a monodromic $\mathcal{D}$-module.

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