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DIFFERENTIABLE SINGULARITIES OF SOLUTIONS OF MICRODIFFERENTIAL EQUATIONS WITH DOUBLE INVOLUTIVE CHARACTERISTICS

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1. INTRODUCTION

We study a class of microdifferential equations with double involutive characteristics in the spaces of micro-distributions.

Let M be a real analytic manifold with complexification X. We consider the operator $P \in \mathcal{E}_X$ of the form $P = P_1 P_2 +$ *lower order* with simple characteristic operators P_j's at a double characteristic point $\dot{q} \in \mathrm{Char}(P_1) \cap \mathrm{Char}(P_2) \cap T^*_M X$. Here P_j's have principal symbols p_j's which are real valued on $T^*_M X$ and satisfy $dp_1 \wedge dp_2 \wedge \omega \neq 0$ for the canonical 1-form ω.

Note that the structure of the microdifferential equation generated by P is completely studied outside of $\mathrm{Char}(P_1) \cap \mathrm{Char}(P_2)$ by M. Sato et al. [6]. On the double characteristic points $\mathrm{Char}(P_1) \cap \mathrm{Char}(P_2)$, N. Tose [7, 8] treats the analytic singularities by using the method of second microlocalization, and G. Uhlmann [9] treats the differentiable singularities by constructing a parametrix.

Under additional assumption of Levi condition, we get

(I) the solvability of the equation

$$Pu = v,$$

in the spaces of micro-distributions.

(II) the relation between the micro-differentiable singularities of micro-distribution solutions of

$$Pu = 0,$$

and the Hamiltonian flows of $\sigma(P_1)$ and $\sigma(P_2)$,

We use Bony-Schapira's calculus (see [2]), the notion of simple sheet operators (see [1]), and the relation between microfunctions with holomorphic parameters and their defining holomorphic functions (see [5] and [4]).

2. Main theorems

Let M be a real analytic manifold with a complexification X, π_M be the projection $T^*_M X \to M = T^*_M M$, and Sp_M be the spectrum map

$$\mathrm{Sp}_M : \pi_M^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M,$$

where $\mathcal{B}_M$ is the sheaf of Sato's hyperfunctions defined on M and $\mathcal{C}_M$ is the sheaf of microfunctions defined on $T^*_M X$.

We introduce the subsheaves $\mathcal{C}^f_M$ and $\mathcal{C}^d_M$ of $\mathcal{C}_M$ by

$$\mathcal{C}^f_M = \mathrm{Sp}_M(\pi_M^{-1}\mathcal{D}'_M),$$
$$\mathcal{C}^d_M = \mathrm{Sp}_M(\pi_M^{-1}C^\infty_M),$$

where $\mathcal{D}'_M$ and C^∞_M are the sheaf of distributions and that of differentiable functions respectively. These sheaves $\mathcal{D}'_M$ and C^∞_M are subsheaves of $\mathcal{B}_M$. Sections of $\mathcal{C}^f_M$ (resp. $\mathcal{C}^d_M$) are called tempered microfunctions (resp. differentiable microfunctions). $\mathcal{C}^f_M$ and $\mathcal{C}^d_M$ are conically soft.

Let V be a regular involutive submanifold in $\dot{T}^*_M X = T^*_M X \setminus T^*_M M$. Then for any $u \in \mathcal{C}^f_M|_V$, we can define the tempered second wave front set $\mathrm{WF}^2_V(u)$ of u along V, which is a closed biconic subset of

$\dot{T}_V\dot{T}^*_M X = T_V\dot{T}^*_M X \setminus T_V V$. Refer to G. Lebeau [3] for the definition. We also define subsheaves $\mathcal{A}^{2,f}_V$ and $\mathcal{A}^{2,d}_V$ of $\mathcal{C}^f_M|_V$ by

$$\mathcal{A}^{2,f}_V(U) = \{u \in \mathcal{C}^f_M|_V(U); \mathrm{WF}^2_V(u) = \emptyset\},$$
$$\mathcal{A}^{2,d}_V(U) = \{u \in \mathcal{C}^d_M|_V(U); \mathrm{WF}^2_V(u) = \emptyset\},$$

for any open subset $U \subset V$.

Let $\dot{q}$ be a point in $\dot{T}^*_M X$. We consider a microdifferential equation

$$Pu = v \tag{2.1}$$

and

$$Pu = 0 \tag{2.2}$$

at $\dot{q}$, where P is a microdifferential operator satisfying the following conditions.

(A.1) There exist two regular involutive submanifolds $V_1^{\mathbb{C}}$ and $V_2^{\mathbb{C}}$ of T^*X, which satisfy

- (A.1.1) the characteristic variety $\mathrm{Char}(P) \subset T^*X$ coincides with $V_1^{\mathbb{C}} \cup V_2^{\mathbb{C}}$,
- (A.1.2) $\mathrm{codim}_{\mathbb{C}}\, V_1^{\mathbb{C}} = \mathrm{codim}_{\mathbb{C}}\, V_2^{\mathbb{C}} = 1$,
- (A.1.3) $V_1^{\mathbb{C}}$ and $V_2^{\mathbb{C}}$ intersect transversally along $V^{\mathbb{C}} = V_1^{\mathbb{C}} \cap V_2^{\mathbb{C}}$, which is a regular involutive submanifold of complex codimension 2 containing $\dot{q}$,
- (A.1.4) P is simple characteristic along $\mathrm{Char}(P) \setminus V^{\mathbb{C}}$, i.e., the principal symbol $p = \sigma(P)$ vanishes exactly up to order 1 along $V_1^{\mathbb{C}} \setminus V^{\mathbb{C}}$ and $V_2^{\mathbb{C}} \setminus V^{\mathbb{C}}$,

(A.1.5) $V_1^{\mathbb{C}}$ and $V_2^{\mathbb{C}}$ are real, i.e., we can take defining functions p_1 of $V_1^{\mathbb{C}}$ and p_2 of $V_2^{\mathbb{C}}$ which are real valued on T_M^*X, and satisfy $dp_1 \neq 0$ and $dp_2 \neq 0$.

(A.2) P satisfies the Levi condition along $V^{\mathbb{C}}$. That is, in this case, the symbol of $(\operatorname{ord} P - 1)$'th order part of P vanishes up to order 1 along $V^{\mathbb{C}}$.

Let us set

$$V_1 = V_1^{\mathbb{C}} \cap T_M^*X,$$
$$V_2 = V_2^{\mathbb{C}} \cap T_M^*X,$$
$$V = V^{\mathbb{C}} \cap T_M^*X.$$

Then, from (A.1.2), (A.1.3) and (A.1.5), we find V_1, V_2 and V are regular involutive submanifolds in T_M^*X of real codimension 1, 1 and 2 respectively.

Let q be a point of V. We denote by b_q the bicharacteristic leaf of V through q, and by b_q^j $(j = 1, 2)$ the bicharacteristic curve of V_j through q. We also set, for a subset $K \subset b_{\dot{q}}$,

$$\hat{K} = \{q \in b_{\dot{q}};\ \text{both } b_q^1 \text{ and } b_q^2 \text{ intersect with } K\}.$$

Our main results are:

Theorem 2.1. *Let P be as above. Then the equation* (2.1) *is solvable for $\mathcal{C}_M^f$ and $\mathcal{C}_M^d$. That is, if v belongs to $\mathcal{C}_{M,\dot{q}}^f$ (resp. $\mathcal{C}_{M,\dot{q}}^d$), there exists a solution u of* (2.1) *belonging to $\mathcal{C}_{M,\dot{q}}^f$ (resp. $\mathcal{C}_{M,\dot{q}}^d$).*

Theorem 2.2. *Let P be as above. There exists a neighborhood $U \subset b_{\dot{q}}$ of $\dot{q}$ with the following property. Assume that φ is a continuous map from the unit square $I \times I$ $(I = [0, 1])$ to U satisfying*

- $\varphi(I, t)$ *is bicharacteristic curve of* V_1 *for any* t,
- $\varphi(s, I)$ *is bicharacteristic curve of* V_2 *for any* s,

and that $u \in \mathcal{C}_M^f(\varphi(I \times I))$ *is a solution of* (2.2). *Then there exist solutions* $u_1 \in \mathcal{A}_{V_1}^{2,f}(\varphi(I \times I))$ *and* $u_2 \in \mathcal{A}_{V_2}^{2,f}(\varphi(I \times I))$ *of* (2.2) *which satisfy*

$$u = u_1 + u_2.$$

Moreover if $\dot{q} \notin \mathrm{WF}(u)$, *the preceding* u_1 *and* u_2 *can be chosen with the property* $\dot{q} \notin \mathrm{WF}(u_1)$ *and* $\dot{q} \notin \mathrm{WF}(u_2)$.

Theorem 2.3. *Let* P *be as above. There exists a neighborhood* $U \subset b_{\dot{q}}$ *of* $\dot{q}$ *with the following property. Assume that* φ *is a continuous map from the unit square* $I \times I$ $(I = [0,1])$ *to* U *satisfying*

- $\varphi(I, t)$ *is bicharacteristic curve of* V_1 *for any* t,
- $\varphi(s, I)$ *is bicharacteristic curve of* V_2 *for any* s,

and that $u \in \mathcal{C}_M^f(\varphi(I \times I))$ *is a solution of* (2.2). *Then we have*

$$\varphi(0,0), \varphi(1,0), \varphi(0,1) \notin \mathrm{WF}(u) \Longrightarrow \varphi(1,1) \notin \mathrm{WF}(u).$$

Corollary 2.4. *Let* P *be as above. There exists a neighborhood* $U \subset b_{\dot{q}}$ *of* $\dot{q}$ *satisfying the following property. Let* K *be an arbitrary connected compact subset of* U *with* $\hat{K} \subset U$. *Then, for any solution* $u \in \mathcal{C}_M^f(\hat{K})$ *of* (2.2), *we have*

$$\mathrm{WF}(u) \cap K = \emptyset \Longrightarrow \mathrm{WF}(u) \cap \hat{K} = \emptyset.$$

3. Goursat problem in complex flat domains

We prove the theorems in the previous section by the well-posedness of the Goursat problem of micro-differential equation in suitable flat domains, by using the Bony-Schapira's actions of micro-differential operators to holomorphic functions. See Bony and Schapira [2].

In this section we assume that $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$ with coordinates

$$x = (x_1, x_2, \ldots, x_n) \in M,$$
$$z = (z_1, z_2, \ldots, z_n) = x + \sqrt{-1}y \in X,$$
$$(x, \sqrt{-1}\xi \cdot dx) \in T^*_M X \text{ with } \xi = (\xi_1, \xi_2, \ldots, \xi_n).$$

We take, using these coordinates, V_j, V and $\dot{q}$ as

$$V_j = \{(x, \sqrt{-1}\xi \cdot dx) \in T^*_M X; \xi_j = 0\} \quad (j = 1, 2),$$
$$V = V_1 \cap V_2 = \{\xi_1 = \xi_2 = 0\},$$
$$\dot{q} = (0; \sqrt{-1}dx_n) \in V.$$

Let $P = D_1D_2 - D_1A - D_2B - C$ be a microdifferential operator defined in a neighborhood of $\dot{q} = (0; \sqrt{-1}dx_n) \in T^*_M X$, where A, B and C are microdifferential operators of order ≤ 0.

We consider the Goursat problem

$$\text{(3.1)} \qquad \begin{cases} (D_1D_2 - D_1A_\Sigma - D_2B_\Sigma - C_\Sigma)f = g \\ f|_{z_1=0} = 0, \quad f|_{z_2=0} = 0 \end{cases}$$

for a given function $g \in \mathcal{O}(\Omega)$ and an unknown function $f \in \mathcal{O}(\Omega)$, where Σ is a hyperplane $\{z_n = \dot{\sigma}\}$, Ω is a suitable flat domain, and A_Σ, B_Σ and C_Σ are the Bony-Schapira's action of A, B and C.

Moreover, for a holomorphic function $f \in \mathcal{O}(\Omega)$, non-negative integer ℓ, and an arbitrary positive valued function λ defined in Ω depending only on z_n, we set

$$d'(z) = \inf_{(\tilde{z}',z'',z_n)\notin\Omega} \sup_{i=1,2} |\tilde{z}_i - z_i|,$$

$$d''(z) = \inf_{(z',\tilde{z}'',z_n)\notin\Omega} \sup_{3\le i\le n-1} |\tilde{z}_i - z_i|,$$

$$\hat{\lambda}(z_n) = \sup_{0\le t\le 1} \lambda(tz_n + (1-t)\dot{\sigma}),$$

$$\|f\|_\lambda = \sup_{z\in\Omega} \frac{|f(z)|}{\hat{\lambda}(z_n)d'(z)^{-1}d''(z)^{-1}},$$

$$\|f\|_{\lambda,\ell} = \sup_{|\alpha|+\alpha_n\le\ell} \|D^\alpha f\|_\lambda.$$

Theorem 3.1. *Let P be as above, Ω be sufficiently small, and λ be an arbitrary positive valued function defined in Ω depending only on z_n. Then the Goursat problem* (3.1) *is well-posed for class G_λ and G_λ^∞. Here $G_\lambda(\Omega)$ and $G_\lambda^\infty(\Omega)$ are vector spaces defined by*

$$G_\lambda(\Omega) = \{f \in \mathcal{O}(\Omega); \|f\|_\lambda < \infty\},$$

$$G_\lambda^\infty(\Omega) = \{f \in \mathcal{O}(\Omega); \|f\|_{\lambda,\ell} < \infty \text{ for any } \ell\}.$$

In the proof of this theorem, we use the notion of simple sheet operators due to Bony [1]. We omit the details.

4. Proof of the results

In this section, we give a sketch of proof of our results.

By a suitable real (quantized) contact transformation, we can reduce our problem to the case

$$M = \mathbb{R}^n \hookrightarrow X = \mathbb{C}^n,$$

$$V_1 = \{\xi_1 = 0\}, V_2 = \{\xi_2 = 0\}, V = \{\xi_1 = \xi_2 = 0\},$$

$$\dot{q} = (0; \sqrt{-1}dx_n).$$

Moreover from the assumption of Levi condition, we deduce

$$P = Q(D_1D_2 - D_1A - D_2B - C)$$

with an elliptic operator Q and operators A, B, C of order ≤ 0. Taking account that an elliptic operator operates bijectively on $\mathcal{C}^f_M$, $\mathcal{C}^d_M$ and $\mathcal{A}^{2,*}_{\#}$, we may assume, from the beginning, that

$$P = D_1D_2 - D_1A - D_2B - C.$$

The solvability (Theorem 2.1) and the decomposition of solutions of homogeneous equations (the first half of Theorem 2.2) is a easy consequence of solvability of the Goursat problem in the previous section.

The last half of Theorem 2.2 can be proved by using the uniqueness of the Goursat problem.

Now we give the proof of Theorem 2.3. From Theorem 2.2, we can take u_j's ($j = 1, 2$) with

$$\begin{gathered} u = u_1 + u_2, \\ u_j \in \mathcal{A}^{2,f}_{V_j}(L) \quad (j = 1, 2), \\ \varphi(0,0) = (0; \sqrt{-1}dx_n) \notin \mathrm{WF}(u_j) \quad (j = 1, 2). \end{gathered}$$

We give a remark that each u_j is a restriction of a tempered microfunction with a holomorphic parameter z_j to $\{\operatorname{Im} z_j = 0\}$. Thus, from the propagation theorem of differentiable singularities along holomorphic parameters (refer to [1]. See also [4]), we have

$$\begin{gathered} \varphi(1,0) \notin \mathrm{WF}(u_1), \\ \varphi(0,1) \notin \mathrm{WF}(u_2). \end{gathered}$$

Since these two points are also outside of $\mathrm{WF}(u)$, the estimates

$$\varphi(1,0) \notin \mathrm{WF}(u-u_1) = \mathrm{WF}(u_2),$$

and

$$\varphi(0,1) \notin \mathrm{WF}(u-u_2) = \mathrm{WF}(u_1),$$

follow. Applying the propagation theorem again, we get

$$\varphi(1,1) \notin \mathrm{WF}(u_2),$$

$$\varphi(1,1) \notin \mathrm{WF}(u_1),$$

which concludes the desired result.

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