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EXISTENCE OF A RECURRENT TRAJECTORY FOR RECURRENT PROCESSES

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Abstract

For a process on a complete metric space, the existence of a recurrent trajectory is discussed. Our results extend some of the known results on recurrent motions in dynamical systems to nonautonomous systems.

1 Introduction

The concept of the skew product flow of the process was first introduced by Dafermos (cf. [2]) for discussing the LaSalle's invariant principle. This concept is more general than Miller and Sell's skew product flow (cf. [8, 9]), because the skew product flow of the process is determined by the property of only the solution of a given equation, that is, do not depend on a equation but its phase space. The study of the skew product flow of the process with respect to do invariant principle has been developed by many authors and many papers (cf. [3, 8]). Furthermore, Hale[3] has applied the skew product flow of the process to stability properties and the existence of the periodic trajectory of periodic process.

In this paper, we shall discuss existence theorems in the global sense for the skew product flow of the process, that is, the existence of a recurrent trajectory of a recurrent process, where the recurrent process means the compact minimal set in the sense of Birkhoff (cf. [10]).

2 Construction of Dynamical Systems and Recurrent Processes

Let R, R^+ and N be denoted by real, nonnegative real and natural numbers, respectively. Let (Y, η) be a complete metric space. A family of mappings $\pi(t), t \in R,: Y \to Y$, is said to be a dynamical system on Y if the following conditions are satisfied :

(D1) $\pi(0)x = x$ for every $x \in Y$;

(D2) $\pi(t)\pi(s)x = \pi(t+s)x$ for all $t, s \in R$ and $x \in Y$;

(D3) $\pi(t)x$ is continuous in $(t, x) \in R \times Y$.

Set $\gamma_\pi(x) = \{\pi(t)x \mid t \in R\}$ and denote by $H_\pi(x)$ the closure of $\gamma_\pi(x)$. Furthermore, set $\omega_\pi(x) = \{y \in Y \mid \eta(y, \pi(t_n)x) \to 0$ for some sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty\}$. For a fixed $x \in Y$, a mapping $\pi(t)x : R \to Y$ is called the motion through x.

A motion $\pi(t)x$ is said to be recurrent if for all $\varepsilon > 0$ there exists an $L_\pi(\varepsilon) > 0$ such that for any $t \in R$ and any interval I of length L_π there is an $s \in I$ such that $\eta(\pi(t)x, \pi(s)x) < \varepsilon$.

A nonempty subset A of Y is invariant if for all $x \in A$, $\pi(t)x$ is in A for any $t \in R$.

A subset Σ of Y is minimal if it is nonempty, closed, invariant and has no such a proper subset.

The following results are known (cf. [10]).

Proposition 2.1 (1) *If $\gamma_\pi(x)$ is precompact in Y, then $H_\pi(x)$ and $\omega_\pi(x)$ are nonempty, compact and invariant.*

(2) *Every nonempty, compact and invariant set contains a minimal set.*

The following important proposition of Birkhoff (cf. [10]) gives the relationship between minimal sets and recurrent motions.

Proposition 2.2 *$\pi(t)x$ is a recurrent motion in Y if and only if $H_\pi(x)$ is a compact minimal set.*

Let (X, δ) be a complete metric space. We introduce the concept of the process which was defined by Dafermos (cf. [2]).

Definition 2.1 *A mapping $u : R^+ \times R \times X \to X$ is a process on X if*

(P1) *$u(0, s, x) = x$ for all $s \in R$, $x \in X$;*

(P2) *$u(t+\tau, s, x) = u(t, \tau + s, u(\tau, s, x))$ for all $t, \tau \in R^+$, $s \in R$, $x \in X$;*

(P3) *u is continuous in $(t, s, x) \in R^+ \times R \times X$.*

Denote by W the space of processes on X with some meric d_1.

A family of mappings $\sigma_1(\tau), \tau \in R$, is defined by

$$\sigma_1(\tau)u = u^\tau$$

for $u \in W$, where $u^\tau(t, s, x) = u(t, s + \tau, x)$ for $(t, s, x) \in R^+ \times R \times X$. Then $\sigma_1(\tau), \tau \in R$, maps from W into W.

Then $\sigma_1(0)u = u$ and $\sigma_1(\tau + \mu)u = u^{\tau+\mu} = \sigma_1(\tau)u^\mu = \sigma_1(\tau)\sigma_1(\mu)u$ for $u \in W$. Hence $\sigma_1(\tau), \tau \in R$, satisfies Axioms (D1) and (D2) of a dynamical system. But we cannot expect that $\sigma_1(\tau), \tau \in R$, on W satisfies the condition (D3). If we restrict W to some subset V, we expect that $\sigma_1(\tau), \tau \in R$, satisfies it. So we give the following hypothesis.

(H1) There is a subset $V \subset W$ which, with some metric d_1 of W, forms a complete metric space and $\sigma_1(\tau), \tau \in R$, is a dynamical system on V.

For the dynamical system $\sigma_1(\tau), \tau \in R$, on V, we again define the following notations. Suppose that $u \in V$. Set $\gamma_{\sigma_1}(u) = \{\sigma_1(t)u \in V \mid t \in R\}$ and denote by $H_{\sigma_1}(u)$ the closure of $\gamma_{\sigma_1}(u)$. The set $H_{\sigma_1}(u)$ is called the hull of u. Also, set $\omega_{\sigma_1}(u) = \{v \in V \mid d_1(v, \sigma_1(t_n)u) \to 0$ for some sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty\}$.

We shall give the definition of a recurrent process.

Definition 2.2 *$u \in V$ is said to be recurrent if $H_{\sigma_1}(u)$ is a compact minimal set.*

The following gives some property of recurrent processes.

Lemma 2.1 *Suppose that* (H1) *is satisfied. If $u \in V$ is recurrent, then every $v \in H_{\sigma_1}(u)$ is a recurrent process.*

Proof. From Proposition 2.2, $\sigma_1(t)v$ is a recurrent motion for every $v \in H_{\sigma_1}(u)$. Because of completeness of V and Proposition 2.2, $H_{\sigma_1}(v)$ is a compact minimal. Namely, v is a recurrent process.

Proposition 2.3 *Suppose that* (H1) *is satisfied. If $u \in V$ is recurrent, then $u \in H_{\sigma_1}(v)$ for every $v \in H_{\sigma_1}(u)$.*

Proof. We assume that u is a recurrent process on X and $v \in H_{\sigma_1}(u)$. v is also a recurrent process by Lemma 2.1. So $H_{\sigma_1}(u)$ and $H_{\sigma_1}(v)$ are compact minimal sets. Since $H_{\sigma_1}(u)$ is invariant, $\gamma_{\sigma_1}(v) \subset H_{\sigma_1}(u)$. Hence, $H_{\sigma_1}(u) = H_{\sigma_1}(v)$. Since $u \in H_{\sigma_1}(u)$, $u \in H_{\sigma_1}(v)$.

Corollary 2.1 *Suppose that* (H1) *is satisfied. If $u \in V$ is recurrent, then $u \in \omega_{\sigma_1}(v)$ for every $v \in \omega_{\sigma_1}(u)$.*

Proof. Since $H_{\sigma_1}(u)$ is a compact minimal set, $\omega_{\sigma_1}(u)$ is nonempty, compact and invariant by using Proposition 2.1 (1). $\omega_{\sigma_1}(u) \subset H_{\sigma_1}(u)$ implies $\omega_{\sigma_1}(u) = H_{\sigma_1}(u)$. Similarly $\omega_{\sigma_1}(v) = H_{\sigma_1}(v)$. This completes the proof of Corollary 2.1 by Proposition 2.3.

3 Skew Product Flows of Processes and The Existence of A Recurrent Trajectory

We construct a skew product flow of the process $u \in V$.

Let (Y, η) be a complete metric space.

Definition 3.1 *A family of mappings $T(t), t \geq 0, : Y \to Y,$ is said to be a C^0-semigroup provided that*

(S1) $T(0)x = x$ *for all* $x \in Y$;

(S2) $T(t)T(s)x = T(t+s)x$ *for all* $t, s \in R^+$, $x \in Y$;

(S3) $T(t)x$ *is continuous in* $(t, x) \in R^+ \times Y$.

Set $\gamma_T(x) = \{T(t)x \mid t \in R^+\}$ and denote by $H_T(x)$ the closure of $\gamma_T(x)$. Furthermore, set $\omega_T(x) = \{y \in Y \mid \eta(y, T(t_n)x) \to 0$ for some sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty\}$. A set $S \subset Y$ is said to be invariant if for any $x \in S$, $T(t)x$ is defined on R and $T(t)x \in S$ for every $t \in R$.

The following result is known (cf. [3]).

Proposition 3.1 *If $\gamma_T(x)$ is precompact in Y, then $\omega_T(x)$ is nonempty, compact and invariant.*

We give the following hypothesis.

(H2) There is a $u \in W$ such that $H_{\sigma_1}(u)$, with some metric d_1 of W, forms a compact metric space and $\sigma_1(\tau), \tau \in R$, is a dynamical system on $H_{\sigma_1}(u)$.

Clearly, if (H2) is satisfied then (H1) is satisfied by setting $V = H_{\sigma_1}(u)$.

A mapping $\rho : (X \times H_{\sigma_1}(u)) \times (X \times H_{\sigma_1}(u)) \to R$ is defined by

$$\rho((x, u), (y, v)) = \delta(x, y) + d_1(u, v).$$

Clearly, $(X \times H_{\sigma_1}(u), \rho)$ is a complete metric space.

Let $\alpha : R^+ \times X \times W \to X$ be defined by

$$\alpha(t, x, u) = u(t, 0, x)$$

for $(t,x,u) \in R^+ \times X \times W$. Also, let $\pi_1(t), t \geq 0, : X \times H_{\sigma_1}(u) \to X \times H_{\sigma_1}(u)$, be defined by

$$\pi_1(t)(x,v) = (\alpha(t,x,v), \sigma_1(t)v)$$

for $(x,v) \in X \times H_{\sigma_1}(u)$.

If $\pi_1(t) : X \times H_{\sigma_1}(u) \to X \times H_{\sigma_1}(u)$ is a C^0-semigroup, we refer to $\pi_1(t), t \geq 0$, as the skew product flow of the process v on X.

Proposition 3.2 *Suppose that* (H2) *is satisfied. Then* $\pi_1(t)$, $t \geq 0$, *is a* C^0*-semigroup on* $X \times H_{\sigma_1}(u)$. *That is,* $\pi_1(t), t \geq 0$, *is the skew product flow of the process* v *on* X.

Proof. $\pi_1(0)(x,v) = (\alpha(0,x,v), \sigma_1(0)v) = (x,v)$ for all $(x,v) \in X \times H_{\sigma_1}(u)$. And

$$\begin{aligned}
\pi_1(t)\pi_1(s)(x,v) &= \pi_1(t)(v(s,0,x), \sigma_1(s)v) \\
&= (\alpha(t, v(s,0,x), \sigma_1(s)v), \sigma_1(t)\sigma_1(s)v) \\
&= (v(t,s,v(s,0,x)), \sigma_1(t)\sigma_1(s)v) \\
&= (v(t+s,0,x), \sigma_1(t)\sigma_1(s)v) \\
&= (\alpha(t+s,x,v), \sigma_1(t+s)v) \\
&= \pi_1(t+s)(x,v).
\end{aligned}$$

Therefore $\pi_1(t), t \geq 0$, satisfies Axioms (S1) and (S2) of a C^0-semigroup.

By the hypothesis (H2), $\sigma_1(t)v$ is continuous, and hence we only show that α is continuous in $(t,x) \in R^+ \times X$. From the condition (P3) of a process, $\alpha(t,x,v) = v(t,0,x)$ is continuous. Consequently $\pi_1(t), t \geq 0$, satisfies the condition (S3). This completes the proof of Proposition 3.2.

For the skew product flow $\pi_1(t), t \geq 0$, we give the following notations. For $(x,v) \in X \times H_{\sigma_1}(u)$, set $\gamma_{\pi_1}(x,v) = \{\pi_1(t)(x,v) \mid t \in R^+\}$ and denote by $H_{\pi_1}(x,v)$ the closure of $\gamma_{\pi_1}(x,v)$. Furthermore, set $\omega_{\pi_1}(x,v) = \{(y,v') \in X \times H_{\sigma_1}(u) \mid \rho(\pi_1(t_n)(x,v),(y,v')) \to 0$ for some sequnece $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty\}$.

Lemma 3.1 *Suppose that the condition* (H2) *is satisfied. If there exists an* $x_0 \in X$ *and a compact set* $K \subset X$ *such that* $\alpha(t,x_0,u) \in K$ *for all* $t \geq 0$, *then* $\pi_1(t)$ *defines a dynamical system on* $\omega_{\pi_1}(x_0,u)$.

Proof. We assume that $(z,w) \in \gamma_{\pi_1}(x_0,u)$. Then from the definition of $\pi_1(t)$, there is a $\tau \in R^+$ such that $\pi_1(\tau)(x_0,u) = (z,w)$. Since

$$\pi_1(\tau)(x_0,u) = (\alpha(\tau,x_0,u), \sigma_1(\tau)u) \in K \times H_{\sigma_1}(u),$$

$\gamma_{\pi_1}(x_0,u) \subset K \times H_{\sigma_1}(u)$. By the hypothesis, $\gamma_{\pi_1}(x_0,u)$ is relatively compact in $X \times H_{\sigma_1}(u)$. Consequently by Proposition 3.1, $\omega_{\pi_1}(x_0,u)$ is a compact invariant set.

For any $(y,v) \in \omega_{\pi_1}(x_0,u)$, there is a sequence $\{t_n\} \subset R^+, t_n \to \infty$ as $n \to \infty$, such that $\pi_1(t_n)(x_0,u) \to (y,v)$. By (P2) and (P3), we have for all $t \in R^+$

$$\begin{aligned} \pi_1(t+t_n)(x_0,u) &= (\alpha(t+t_n,x_0,u), \sigma_1(t+t_n)u) \\ &= (u(t+t_n,0,x_0), \sigma_1(t)\sigma_1(t_n)u) \\ &= (u(t,t_n,u(t_n,0,x_0)), \sigma_1(t)\sigma_1(t_n)u) \\ &= ((\sigma_1(t_n)u)(t,0,u(t_n,0,x_0)), \sigma_1(t)\sigma_1(t_n)u) \to (v(t,0,y), \sigma_1(t)v) \end{aligned}$$

as $n \to \infty$. Hence, $\pi_1(t), t \geq 0$, is well defined on $\omega_{\pi_1}(x_0,u)$. Since $\omega_{\pi_1}(x_0,u)$ is compact invariant, $\pi_1(t)$ is defined on R and $\pi_1(t), t \in R$, is a dynamical system on $\omega_{\pi_1}(x_0,u)$. This completes the proof of Lemma 3.1.

Next we define a recurrent trajectory. To do this, we introduce the compact open topology on $C(R,X)$, where $C(R,X)$ is the space of continuous functions on R into X. Suppose that $u \in W$ and $x' \in X$ are fixed and that $\alpha(t,x',u)$ is in $C(R,X)$. A mapping $\sigma_2(\tau), \tau \in R$, is defined by

$$\sigma_2(\tau)\alpha = \alpha_\tau$$

where $\alpha_\tau(t,x',u) = \alpha(t+\tau,x',u)$ for $t \in R$. For $\alpha \in C(R,X)$ set $\gamma_{\sigma_2}(\alpha) = \{\sigma_2(\tau)\alpha \mid \tau \in R\}$ and denote by $H_{\sigma_2}(\alpha)$ the closure of $\gamma_{\sigma_2}(\alpha)$ by the compact open topology.

We also define the following as a metric d_2 on $C(R,X)$. Since R is separable, there is a countable set $K^2_\infty = \{t_1,t_2,\cdots,t_m,\cdots\}$ such that K^2_∞ is dense in R. Then put

$$d_2(\alpha,\beta) = \sum_{m=1}^{\infty} \frac{\delta(\alpha(t_m),\beta(t_m))}{2^m(1+\delta(\alpha(t_m),\beta(t_m)))}$$

for $\alpha,\beta \in C(R,X)$. We denote by $(C(R,X),d_2)$ the space $C(R,X)$ equipped with the metric d_2. If $H_{\sigma_2}(\alpha)$ is compact in $C(R,X)$, then it is equicontinuous (cf. [4, Theorem 1.4, P.217]). Since R is separable, the topologies of $H_{\sigma_2}(\alpha)$ induced from $C(R,X)$ and $(C(R,X),d_2)$ are equivalent to each other (cf. [4, Proposition 1.3, P.216]). Hence, $H_{\sigma_2}(\alpha)$ is metrizable by d_2.

Lemma 3.2 *Suppose that $H_{\sigma_2}(\alpha)$ is compact. Then $\sigma_2(t), t \in R$, is a dynamical system on $H_{\sigma_2}(\alpha)$.*

Proof. Clearly $\sigma_2(t), t \in R$, maps $H_{\sigma_2}(\alpha)$ to $H_{\sigma_2}(\alpha)$. Therefore we must check the condition (D1), (D2) and (D3) of a dynamical system.

$\sigma_2(0)\beta = \beta$ for all $\beta \in H_{\sigma_2}(\alpha)$. And $\sigma_2(t)\sigma_2(s)\beta = \sigma_2(t)\beta_s = \beta_{t+s} = \sigma_2(t+s)\beta$ for $\beta \in H_{\sigma_2}(\alpha)$. Therefore $\sigma_2(t), t \in R$, satisfies Axioms (D1) and (D2) of a dynamical system.

Suppose that $t \in R$ and $\varepsilon > 0$ are arbitrary. There exists an $n \in N$ such that $\frac{1}{2^n} \leq \frac{\varepsilon}{2}$. Since β is continuous, β is uniformly continuous on any compact set. Consequently there is a $\theta(\varepsilon,t) > 0$ such that if $\mid t-s \mid < \theta$ then

$$\mid \beta(t+t_m) - \beta(s+t_m) \mid_X < \frac{\varepsilon}{2}$$

for all $m \leq n$, where $t_m \in K^2_\infty$. Hence,

$$\begin{aligned} d_2(\sigma_2(t)\beta, \sigma_2(s)\beta) &= \sum_{j=1}^{n} \frac{|(\sigma_2(t)\beta)(t_j) - (\sigma_2(s)\beta)(t_j)|_X}{2^j(1+|(\sigma_2(t)\beta)(t_j) - (\sigma_2(s)\beta)(t_j)|_X)} \\ &+ \sum_{j=n+1}^{\infty} \frac{|(\sigma_2(t)\beta)(t_j) - (\sigma_2(s)\beta)(t_j)|_X}{2^j(1+|(\sigma_2(t)\beta)(t_j) - (\sigma_2(s)\beta)(t_j)|_X)} \\ &< \sum_{j=1}^{\infty} \frac{\frac{\varepsilon}{2}}{2^j} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{\varepsilon}{2} + \frac{1}{2^n} < \varepsilon. \end{aligned}$$

Therefore $\sigma_2(t), t \in R$, satisfies Axiom (D3). This completes the proof of Lemma 3.2.

Definition 3.2 *Assume $u \in W$ and $x' \in X$. $\alpha(t, x', u)$ is said to be a recurrent trajectory through $(0, x')$ if $\alpha(t, x', u)$ is defined on R and $H_{\sigma_2}(\alpha)$ is compact minimal.*

Lemma 3.3 *Suppose that $u \in W$ and $x' \in X$ are fixed. $\alpha(t, x', u)$ is a recurrent trajectory through $(0, x')$ if it is a recurrent fuction. That is, for every $\varepsilon > 0$ and any compact $S \subset R$ there is an $L_\alpha(\varepsilon, S) > 0$ such that for all $t \in R$ and any interval $I \subset R$ of length L_α there exists an $s \in I$ such that*

$$\delta(\alpha(\tau + t, x', u), \alpha(\tau + s, x', u)) < \varepsilon$$

for every $\tau \in S$.

Proof. Suppose that α is a recurrent function. For any $\varepsilon > 0$, there exists an $n_0 \in N$ such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$ for every $n > n_0$. Set $I_n = \{t_j \in K^2_\infty \mid j \leq n\}$. Since I_n is compact, there is an $L_\alpha(\frac{\varepsilon}{2}, I_n) > 0$ such that for every $t \in R$ and any interval $I \subset R$ of length L_α there is an $s \in I$ such that

$$\delta(\alpha(t + t_j, x', u), \alpha(s + t_j, x', u)) < \frac{\varepsilon}{2}$$

for all $t_j \in I_n$. That is, for any $t_j \in I_n$,

$$\delta((\sigma_2(t)\alpha)(t_j, x', u), (\sigma_2(s)\alpha)(t_j, x', u)) < \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned} d_2(\sigma_2(t)\alpha, \sigma_2(s)\alpha) &= \sum_{j=1}^{n} \frac{\delta((\sigma_2(t)\alpha)(t_j, x', u), (\sigma_2(s)\alpha)(t_j, x', u))}{2^j(1 + \delta((\sigma_2(t)\alpha)(t_j, x', u), (\sigma_2(s)\alpha)(t_j, x', u)))} \\ &+ \sum_{j=n+1}^{\infty} \frac{\delta((\sigma_2(t)\alpha)(t_j, x', u), (\sigma_2(s)\alpha)(t_j, x', u))}{2_j(1 + \delta((\sigma_2(t)\alpha)(t_j, x', u), (\sigma_2(s)\alpha)(t_j, x', u)))} \\ &< \sum_{j=1}^{\infty} \frac{\frac{\varepsilon}{2}}{2^j} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{\varepsilon}{2} + \frac{1}{2^n} < \varepsilon. \end{aligned}$$

Conseqently, $\sigma_2(t)\alpha$ is a recurrent motion. By Proposition 2.2, $\alpha(t, x', u)$ is a recurrent trajectory. This completes the proof of Lemma 3.3.

Now, we shall give the existence theorem of a recurrent trajectory.

Theorem 3.1 *Suppose that the condition* (H2) *is satisfied. If there exists an $x_0 \in X$ and a compact set $K \subset X$ such that $\alpha(t, x_0, u) \in K$ for all $t \geq 0$, then there is a $(y, v) \in \omega_{\pi_1}(x_0, u)$ such that v is a recurrent process and $\alpha(t, y, v)$ is a recurrent trajectory through* $(0, y)$.

Proof. From Lemma 3.1, $\pi_1(t), t \in R$, is a dynamical system on $\omega_{\pi_1}(x_0, u)$. Hence, by Proposition 2.1 (2), there is a compact minimal set $\Sigma \subset \omega_{\pi_1}(x_0, u)$. Suppose that (y, v) is in Σ. By Proposition 2.2, $\pi_1(t)(y, v)$ is a recurrent motion. That is, for every $\varepsilon > 0$ there exists an $L_{\pi_1}(\varepsilon) > 0$ such that for all $t \in R$ and any interval $I \subset R$ of length L_{π_1} there is an $s \in I$ such that $\rho(\pi_1(t)(y, v), \pi_1(s)(y, v)) < \varepsilon$. From the definition of ρ,

$$\delta(\alpha(t, y, v), \alpha(s, y, v)) + d_1(\sigma_1(t)v, \sigma_1(s)v) < \varepsilon.$$

Especially,

$$d_1(\sigma_1(t)v, \sigma_1(s)v) < \varepsilon.$$

Consequently, $\sigma_1(t)v$ is a recurrent motion. Then, by Proposition 2.2, $H_{\sigma_1}(v)$ is compact minimal because of completeness of (V, d_1). Hence v is a recurrent process.

We assume that $S \subset R$ is any compact interval. Then by the recurrence of $\pi_1(y, v)$, for any $\tau \in S$

$$\delta(\alpha(t + \tau, y, v), \alpha(s + \tau, y, v)) < \varepsilon.$$

By Lemma 3.3, $\alpha(t, y, v)$ is a recurrent trajectory through $(0, y)$. This completes the proof of Theorem 3.1.

We will obtain the following main theorem.

Theorem 3.2 *Suppose that the condition* (H2) *is satisfied. If there exists a recurrent process u and that there is an $x_0 \in X$ and a compact set $K \subset X$ such that $\alpha(t, x_0, u) \in K$ for all $t \geq 0$, then there is a $z \in X$ such that $\alpha(t, z, u)$ is a recurrent trajectory through* $(0, z)$.

Proof. By Theorem 3.1, there is a $(y, v) \in \omega_{\pi_1}(x_0, u)$. From the definition of $\pi_1(t)$,

$$\alpha(t_n, x_0, u) = u(t_n, 0, x_0) \to y \ , \ \sigma_1(t_n)u \to v$$

for some sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$. Since u is a recurrent process and $v \in \omega_{\sigma_1}(u)$, $u \in \omega_{\sigma_1}(v)$ by Corollary 2.1. Conseuently, $\sigma_1(s_n)v \to u$ as $n \to \infty$ for some sequence $\{s_n\}$.

Since $(y,v) \in \Sigma$, $\pi_1(t)(y,v)$ is a recurrent motion from Proposition 2.2 and $H_{\pi_1}(y,v)$ is compact minimal by Proposition 2.2. Hence by using Proposition 2.1 (1), $\omega_{\pi_1}(y,v)$ is a compact minimal set. There is a subsequence $\{s'_n\}$ of $\{s_n\}$ and a $z \in K$ such that

$$\begin{aligned}\pi(s'_n)(y,v) &= (\alpha(s'_n,y,v),\sigma_1(s'_n)v) \\ &= (v(s'_n,0,y),\sigma_1(s'_n)v) \to (z,u) \in \omega_{\pi_1}(y,v)\end{aligned}$$

as $s'_n \to \infty$. Since $\omega_{\pi_1}(y,v)$ is compact minimal, $\pi_1(t)(z,u)$ is a recurrent motion by Proposition 2.2. Similarly to the proof of Theorem 3.1, $\alpha(t,z,u)$ is a recurrent trajectory through $(0,z)$. This completes the proof of Theorem 3.2.

Examples are given elsewhere.

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